Category Theory

31.01.2025

TODAY

Initial, Terminal objects and Products

RECAP

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}
\qquad
\begin{tikzcd} A \arrow[loop below, "id_A"] \end{tikzcd}
\qquad
\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, bend left, "g"] & C \end{tikzcd}$$

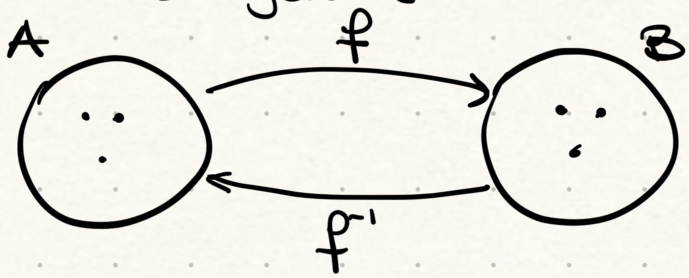
$f;g$

$$A \cong B \iff \exists f: A \longrightarrow B.\ g: B \longrightarrow A.$$
$$g \circ f = id_B \wedge f \circ g = id_A$$

In Set $\quad A \cong B$ is a bijection

$\downarrow$

$|A| = |B|$



: surjective

: injective

In Pos

Posets Partial Order Sets

a morphism in Pos is a monotone function

$$f: (A, \leq_A) \longrightarrow (B, \leq_B)$$

$x, y \in A \qquad x \leq_A y \implies f(x) \leq_B f(y)$

Initial and Terminal Objects

$$\frac{}{\Gamma \vdash \mathrm{True}} \qquad\qquad \frac{}{\Gamma \vdash () : \mathrm{unit}}$$

$(A, \leq)$ the top element $\top$ is an element s.t. for all $x \in A$. $x \leq \top$

The terminal object (denoted by $\mathbb{1}$) is an object of of a category $\mathbb{C}$ s.t for all objects $X \in \mathbb{C}$

$$\exists! \quad X \dashrightarrow \mathbb{1}$$

$$\frac{}{\mathrm{False} \vdash A} \qquad\qquad \frac{}{t_1 : \emptyset \vdash t_2 : A}$$

$$\bot \leq a \quad \forall a \in (X, \leq)$$

The initial object denoted by $\emptyset$

s.t $\forall X \in \mathbb{C}. \exists! \emptyset \dashrightarrow A$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \text{ Intro} \qquad \frac{\Gamma \vdash t_1 : A \quad \Gamma \vdash t_2 : B}{\Gamma \vdash \langle t_1, t_2 \rangle : A \times B}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \text{ Elim} \qquad \frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash \pi_1(t) : A} \quad \frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash \pi_2(t) : B}$$

$(A, \leq)$ called 'meet'

$$\begin{tikzcd}
x \arrow[dr, dash, "\leq"] & & y \arrow[dl, dash, "\leq"'] \\
& x \wedge y \arrow[d, dash, "\leq"] & \\
& z \arrow[uul, dash, bend left, "\leq"] \arrow[uur, dash, bend right, "\leq"'] &
\end{tikzcd}$$

$x \wedge y$ is called "Greatest Lower Bound"

Given x, y in A

- $\exists\, x \wedge y$. s.t. $x \wedge y \leq x$ and $x \wedge y \leq y$ — "it is a lower bound"
- $\forall z \in A.\ z \leq x$ and $z \leq y$ — "for all other lower bounds"
- $z \leq x \wedge y$ — "$x \wedge y$ is the greatest"

The meaning of commuting diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

means $k \circ h = g \circ f$

Global Elements

$$1 \xrightarrow{x} A$$

In Set, $1 \xrightarrow{x} A$ is equivalent to

" $x \in A$ "

Generalised Elements

$$Z \longrightarrow A$$

Product

$$\begin{tikzcd}
X & X \times Y \arrow[l, "\pi_x"'] \arrow[r, "\pi_y"] & Y \\
& Z \arrow[ul, "x"] \arrow[u, dashed, "\langle x, y \rangle"] \arrow[ur, "y"'] &
\end{tikzcd}$$

β-rules of λ-calculus

$$\boxed{\begin{aligned} \pi_x \cdot \langle x, y \rangle &= x \\ \pi_y \cdot \langle x, y \rangle &= y \end{aligned}}$$

η-rules of λ-calculus

$$\boxed{\langle \pi_1 \cdot h, \pi_2 \cdot h \rangle = h}$$

(uniqueness property)

Explanation:

Given objects X, Y in $\mathbb{C}$

$\exists$ a "UNIVERSAL" object

$$\boxed{X \xleftarrow{\pi_x} X \times Y \xrightarrow{\pi_y} Y}$$

such that for any other object

$$\boxed{X \xleftarrow{x} Z \xrightarrow{y} Y}$$

$\exists!\ Z \overset{\langle x, y \rangle}{\dashrightarrow} X \times Y$. such that the above diagram commutes.

In **Set**

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}$$

$$\pi_1 : A \times B \longrightarrow A$$

$$\pi_2 : A \times B \longrightarrow B$$

$$\langle f, g \rangle : Z \longrightarrow A \times B$$

$$\langle f, g \rangle (z) = (f(z), g(z))$$

Exercise. What is the product in Pos?

Exercise. Prove initial objects terminal objects and products are UNIQUE up-to isomorphism.

Elimination

$$
\begin{tikzcd}
X & X \times Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Y \\
& Z \arrow[ul, "x"] \arrow[u, dotted, "\langle x, y \rangle"'] \arrow[ur, "y"'] &
\end{tikzcd}
$$

Introduction

Introduction

$$
\begin{tikzcd}
X \arrow[r, "inl"] \arrow[dr, "f"'] & X+Y \arrow[d, dotted, "{[f,g]}"] & Y \arrow[l, "inr"'] \arrow[dl, "g"] \\
& Z &
\end{tikzcd}
$$

Elimination

$$
\frac{\Gamma \vdash X \vee Y \qquad \Gamma, X \vdash Z \qquad \Gamma, Y \vdash Z}{\Gamma \vdash Z}
$$

$$
\frac{\Gamma \vdash t : X+Y \qquad \Gamma, x:X \vdash t_1 : Z \qquad \Gamma, y:Y \vdash t_2 : Z}{\Gamma \vdash \text{case}(t) \text{ of } \begin{array}{l} inl(x) \Rightarrow t_1 \\ inr(y) \Rightarrow t_2 \end{array} : Z}
$$

(Elimination of $\vee$)

$$
\frac{\Gamma \vdash X}{\Gamma \vdash X \vee Y} \qquad \frac{\Gamma \vdash Y}{\Gamma \vdash X \vee Y}
$$

(Introduction of $\vee$)

In Set

$$A + B = \{(0, x) \mid x \in A\} \cup \{(1, y) \mid y \in B\}$$

$$\{0,1,2\} \cup \{0,1\} = \{0,1,2\}$$

$$\{0,1,2\} \oplus \{0,1\} = \{(0,0), (0,1), (0,2), (1,0), (1,1)\}$$

left or right

element

NNO (Natural Numbers Object)

Introduction

Elimination

$$
\begin{tikzcd}
\overset{*}{1} \arrow[r, "zero"] \arrow[dr, "base"'] & \overset{n}{\mathbb{N}} \arrow[r, "succ"] \arrow[d, dashed, "{h(base, ind)}"] & \overset{n+1}{\mathbb{N}} \arrow[d, dashed, "{h(base, ind)}"] \\
& Z \arrow[r, "ind"'] & Z
\end{tikzcd}
$$

$$\boxed{\begin{aligned} h \cdot zero &= base \\ h \cdot succ &= ind \cdot h \end{aligned}}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} h(0) &= base(*) \\ h(n+1) &= ind(h(n)) \end{aligned}}$$

$$0 = zero(*)$$

$$n+1 = succ(n)$$

g s.t $\quad g \cdot zero = base$

$\qquad\quad g \cdot succ = ind \cdot g$

then $g = h$

$$h(base, ind) \cdot zero \implies$$

Exercise Prove using the universality property of NNO that every number is either <u>even</u> or <u>odd</u>.

Induction on the Natural Numbers Object

Define a Property $P \subseteq \mathbb{N}$

"the numbers $n \in \mathbb{N}$ such that they satisfy a certain formula φ"

$$\boxed{P = \{ n \in \mathbb{N} \mid \varphi(n) \}}$$

Obviously $P \subseteq \mathbb{N}$ for every formula φ, hence it is a "subobject" of $\mathbb{N}$

$$\begin{array}{ccccc} & & P & & P \\ & & \cap\!| & & \cap\!| \\ 1 & \xrightarrow{\text{zero}} & \mathbb{N} & \xrightarrow[\text{succ}]{} & \mathbb{N} \end{array}$$

If we can give two maps

$$\underbrace{1 \xrightarrow{b} P}_{\text{base}} \quad \text{and} \quad \underbrace{P \xrightarrow{i} P}_{\text{induction}}$$

then by uniqueness property of $\mathbb{N}$, we have $P \cong \mathbb{N}$

$$\begin{tikzcd}
 & P \arrow[r] \arrow[d, phantom, sloped, "\subseteq"] \arrow[dr, phantom, "2"] & P \arrow[d, phantom, sloped, "\subseteq"] \\
1 \arrow[ur] \arrow[r] \arrow[ur, phantom, "1"] & \mathbb{N} \arrow[r] \arrow[u, dashed, bend right] & \mathbb{N} \arrow[u, dashed, bend right]
\end{tikzcd}$$

We prove the subset relation seen as an injective function together with the unique arrow from $\mathbb{N}$ yields $P \cong \mathbb{N}$

We introduce notation

For $1 \xrightarrow{z} Z \xrightarrow{s} Z$

Then $\exists!$ arrow $N \to Z$ which makes the N diagram commute

$$(z, s) : N \longrightarrow Z$$

We first prove $\subseteq \cdot (b, i) = id_N$

$$\begin{tikzcd}
1 \arrow[r, "zero"] \arrow[dr, bend right, "b"'] \arrow[ddr, bend right=50, "zero"'] \arrow[dr, phantom, "1", bend left=10] \arrow[ddr, phantom, "3", bend right=20] & N \arrow[r, "succ"] \arrow[d, dashed, "(b, i)"] \arrow[dr, phantom, "2"] & N \arrow[d, dashed, "(b, i)"] \\
& P \arrow[r, "i"] \arrow[d, "\subseteq"'] \arrow[dr, phantom, "4"] & P \arrow[d, "\subseteq"] \\
& N \arrow[r, "succ"'] & N
\end{tikzcd}$$

$\exists!$ arrow $N \longrightarrow N$ namely $(zero, succ)$ which makes 1,3 and 2,4 commute

but! also $\subseteq \cdot (b, i)$ makes those diagrams commute. Hence

$$\subseteq \cdot (\!| b, i |\!) = (\!| zero, succ |\!)$$

and clearly $(\!| zero, succ |\!) = id_{\mathbb{N}}$ ✓

Next we prove $(\!| b, i |\!) \cdot \subseteq = id_P$

if we can prove

$$\subseteq \cdot (\!| b, i |\!) \cdot \subseteq = \subseteq \cdot id_P$$

we're done because by injectivity of $\subseteq$

$$\subseteq(x) = \subseteq(y) \Rightarrow x = y \quad \forall x, y$$

but $\underbrace{\subseteq \cdot (\!| b, i |\!)}_{= id_{\mathbb{N}}} \cdot \subseteq = \subseteq \cdot id_P$

hence $\subseteq = \subseteq \iff$ True ✓

$\square$

Injectivity Generalised

$$\begin{tikzcd} Z \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & A \arrow[r, hook, "f"] & B \end{tikzcd}$$

monomorphism

$$\boxed{f \circ x = f \cdot y \Rightarrow x = y}$$

In Set

$x \in A$
$y \in A$ and $\boxed{f(x) = f(y) \Rightarrow x = y}$

Example

In Set Define the boolean object

$2 = 1+1$

① True $=$ $inl(*)$

False $=$ $inr(*)$

We can define P in two ways

$P : \mathbb{N} \to 2$

$P(n) = even(n) \vee odd(n)$

$(\vee) : 2 \times 2 \to 2$

$P = \{ n \in \mathbb{N} \mid even(n) \vee odd(n) \}$

Define $\begin{array}{ccc} 1 & \xrightarrow{b} & P \\ * & \longmapsto & \emptyset \end{array}$ and $\begin{array}{ccc} P & \xrightarrow{i} & P \\ n & \longmapsto & n+1 \end{array}$

- We should prove b and i are well-defined, that is, that $b(*)$ and $i(n)$ are either even or odd.
- Then we have to prove that $\subseteq$ makes the triangle and the rectangle commute ① ②

Then by previous exercise we completed the proof.