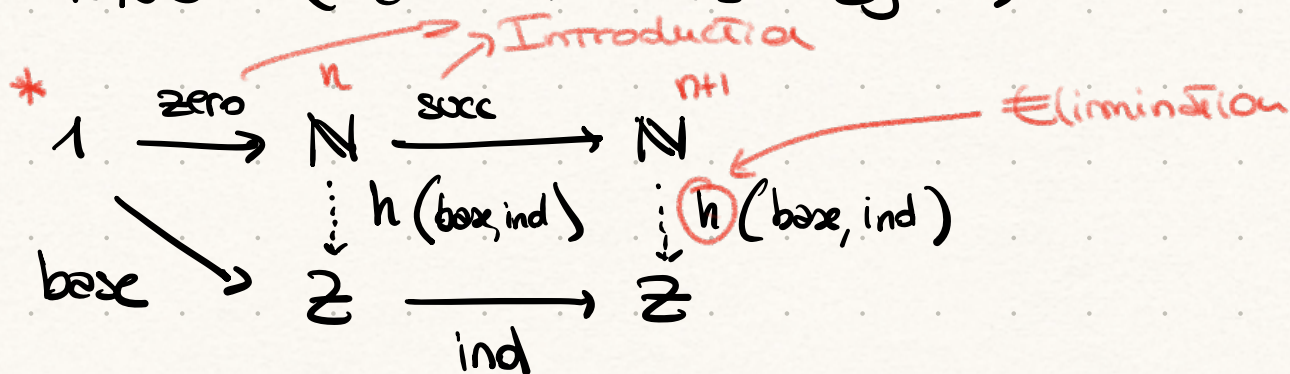


NNO (Natural Numbers Object)



$$\begin{aligned} h \cdot \text{zero} &= \text{base} \\ h \cdot \text{succ} &= \text{ind} \cdot h \end{aligned}$$



$$\begin{aligned} h(0) &= \text{base}(\ast) \\ h(n+1) &= \text{ind}(h(n)) \end{aligned}$$

$$\emptyset = \text{zero}(\ast)$$

$$n+1 = \text{succ}(n)$$

$$g \text{ s.t. } \begin{aligned} g \cdot \text{zero} &= \text{base} \\ g \cdot \text{succ} &= \text{ind} \cdot g \end{aligned}$$

$$\text{then } g = h$$

$$h(\text{base}, \text{ind}) \cdot \text{zero} \Rightarrow$$

Exercise

Prove using the universality property of NNO that every number is either even or odd.

Induction on the Natural Numbers Object

Define a Property $P \subseteq \mathbb{N}$

"the numbers $n \in \mathbb{N}$ such that
they satisfy a certain
formula φ "

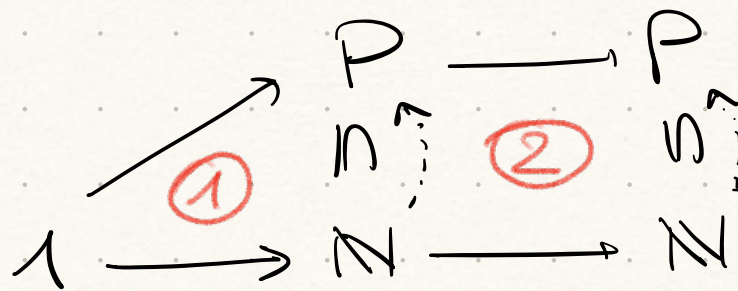
$$P = \{ n \in \mathbb{N} \mid \varphi(n) \}$$

Obviously $P \subseteq \mathbb{N}$ for every
formula φ , hence
it is a "subobject" of \mathbb{N}

$$\begin{array}{ccc} P & & P \\ \text{in} & & \text{in} \\ 1 \xrightarrow{\text{zero}} \mathbb{N} & \xrightarrow{\text{succ}} & \mathbb{N} \end{array}$$

If we can give two maps
 $1 \xrightarrow{b} P$ and $P \xrightarrow{i} P$
↙ ↘
base induction

then by uniqueness property
of \mathbb{N} , we have $P \cong \mathbb{N}$



We Prove the subset
relation seen as an injective
function together with the
unique arrow from \mathbb{N} yields
 $P \cong \mathbb{N}$

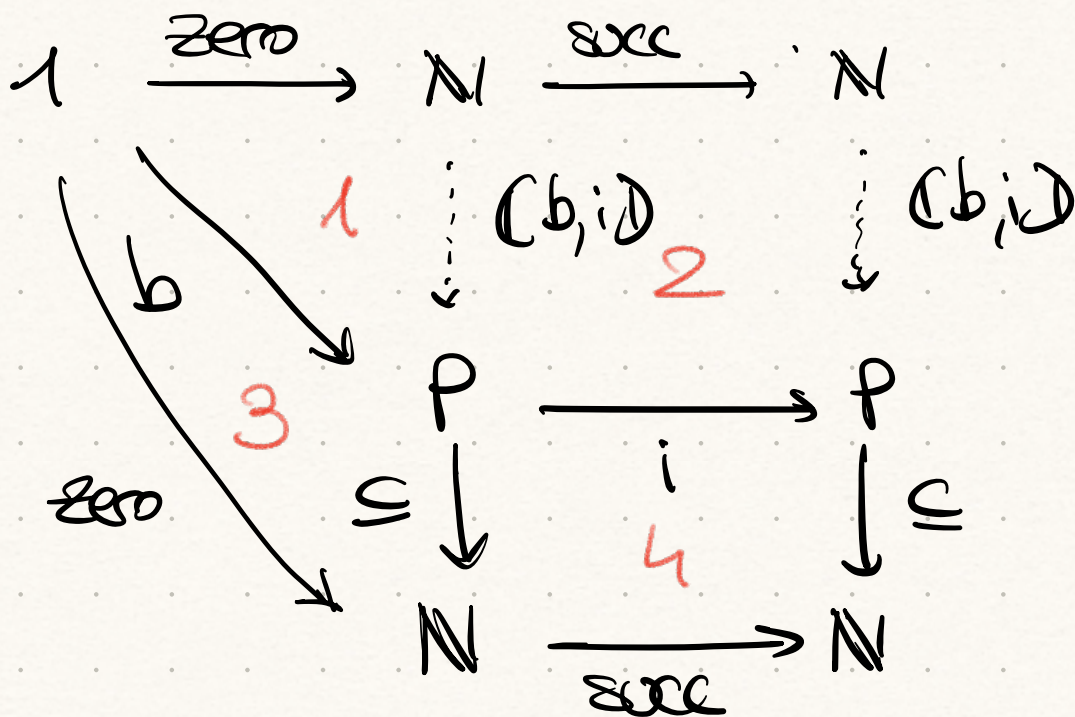
We introduce notation

$$\text{For } 1 \xrightarrow{z} \mathbb{Z} \xrightarrow{s} \mathbb{Z}$$

Then $\exists!$ arrow $\mathbb{N} \rightarrow \mathbb{Z}$ which makes the \mathbb{N} diagram commute

$$(z, s) : \mathbb{N} \longrightarrow \mathbb{Z}$$

We first prove $\varepsilon \cdot (b, i) = \text{id}_{\mathbb{N}}$



$\exists!$ arrow $\mathbb{N} \rightarrow \mathbb{N}$ namely $(\text{zero}, \text{succ})$ which makes $1, 3$ and $2, 4$ commute but! also $\varepsilon \cdot (b, i)$ makes those

diagonal commutative. Hence

$$\subseteq \cdot (b, i) = (\text{zero}, \text{succ})$$

and clearly $(\text{zero}, \text{succ}) = \text{id}_N$ ✓

Next we prove $(b, i) \cdot \subseteq = \text{id}_P$

if we can prove

$$\subseteq \cdot (b, i) \cdot \subseteq = \subseteq \cdot \text{id}_P$$

we're done because by injectivity
of \subseteq

$$\subseteq(x) = \subseteq(y) \Rightarrow x = y \quad \forall x, y$$

but $\subseteq \cdot (b, i) \cdot \subseteq = \subseteq \cdot \text{id}_P$

" id_N "

hence $\subseteq = \subseteq \Leftrightarrow \text{True}$ ✓

□

Injectivity Generalised

$$\begin{array}{c} x \\ \xrightarrow{\quad} \\ y \end{array} A \xrightarrow{f} B \quad \text{monomorphism}$$

$$f \circ x = f \circ y \Rightarrow x = y$$

In Set

$$\begin{array}{l} x \in A \\ y \in A \end{array}$$

and

$$f(x) = f(y) \Rightarrow x = y$$

Example

In Set Define the
boolean object

$$2 = 1 + 1$$

$$\text{True} = \text{inl}(\ast)$$

$$\text{False} = \text{inr}(\ast)$$

We can define P in two ways

$$P: \mathbb{N} \rightarrow 2$$

$$P(n) = \text{even}(n) \vee \text{odd}(n)$$

$$P = \{n \in \mathbb{N} \mid \text{even}(n) \vee \text{odd}(n)\}$$

$$(v): 2 \times 2 \rightarrow 2$$

Define

$$1 \xrightarrow{b} P \quad \text{and} \quad P \xrightarrow{i} P$$
$$\ast \mapsto \emptyset \quad n \mapsto n+1$$

- We should prove b and i are well-defined, that is, that $b(\ast)$ and $i(n)$ are either even or odd.
- Then we have to prove that \subseteq makes the triangle and the rectangle commute ① ②

Then by previous exercise we completed the proof.