NNO (Natural Numbers Object)

Introduction

Elimination

$$
\begin{tikzcd}
\overset{*}{1} \arrow[r, "zero"] \arrow[dr, "base"'] & \overset{n}{\mathbb{N}} \arrow[r, "succ"] \arrow[d, dashed, "h(base, ind)"] & \overset{n+1}{\mathbb{N}} \arrow[d, dashed, "h(base, ind)"] \\
& Z \arrow[r, "ind"'] & Z
\end{tikzcd}
$$

$$\boxed{\begin{aligned} h \cdot zero &= base \\ h \cdot succ &= ind \cdot h \end{aligned}}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} h(0) &= base(*) \\ h(n+1) &= ind(h(n)) \end{aligned}}$$

$$0 = zero(*)$$

$$n+1 = succ(n)$$

g s.t. $g \cdot zero = base$

$g \cdot succ = ind \cdot g$

then $g = h$

$$h(base, ind) \cdot zero \implies$$

Exercise Prove using the universality property of NNO that every number is either even or odd.

Induction on the Natural Numbers Object

Define a Property $P \subseteq \mathbb{N}$

"the numbers $n \in \mathbb{N}$ such that they satisfy a certain formula φ"

$$\boxed{P = \{ n \in \mathbb{N} \mid \varphi(n) \}}$$

Obviously $P \subseteq \mathbb{N}$ for every formula φ, hence it is a "subobject" of $\mathbb{N}$

$$\begin{array}{ccccc} & & P & & P \\ & & \cap\!\shortmid & & \cap\!\shortmid \\ 1 & \xrightarrow{zero} & \mathbb{N} & \xrightarrow[succ]{} & \mathbb{N} \end{array}$$

If we can give two maps

$$1 \xrightarrow{b} P \quad \text{and} \quad P \xrightarrow{i} P$$

base induction

then by uniqueness property of $\mathbb{N}$, we have $P \cong \mathbb{N}$

$$\begin{tikzcd}
 & P \arrow[r] & P \\
1 \arrow[ur] \arrow[r] \arrow[ur, phantom, shift right=3ex, "1"] & \mathbb{N} \arrow[u, phantom, sloped, "\supseteq"] \arrow[u, dashed, bend right] \arrow[r] \arrow[ur, phantom, "2"] & \mathbb{N} \arrow[u, phantom, sloped, "\supseteq"] \arrow[u, dashed, bend right]
\end{tikzcd}$$

We prove the subset relation seen as an injective function together with the unique arrow from $\mathbb{N}$ yields $P \cong \mathbb{N}$

We introduce notation

For $1 \xrightarrow{z} Z \xrightarrow{s} Z$

Then $\exists!$ arrow $N \to Z$ which makes the N diagram commute

$$(z, s) : N \longrightarrow Z$$

We first prove $\subseteq \cdot (b, i) = id_N$

$$\begin{tikzcd}
1 \arrow[r, "zero"] \arrow[dr, bend left, "b"'] \arrow[ddr, bend right=40, "zero"'] & N \arrow[r, "succ"] \arrow[d, dotted, "(b,i)"] & N \arrow[d, dotted, "(b,i)"] \\
& P \arrow[r, "i"] \arrow[d, "\subseteq"'] & P \arrow[d, "\subseteq"] \\
& N \arrow[r, "succ"'] & N
\end{tikzcd}$$

$\exists!$ arrow $N \to N$ namely $(zero, succ)$ which makes 1, 3 and 2, 4 commute

but! also $\subseteq . (b, i)$ makes those diagrams commute. Hence

$$\subseteq \cdot (\!|b, i|\!) = (\!|zero, succ|\!)$$

and clearly $(\!|zero, succ|\!) = id_{\mathbb{N}}$ ✓

Next we prove $(\!|b, i|\!) \cdot \subseteq = id_P$

if we can prove

$$\subseteq \cdot (\!|b, i|\!) \cdot \subseteq = \subseteq \cdot id_P$$

we're done because by injectivity of $\subseteq$

$$\subseteq(x) = \subseteq(y) \implies x = y \quad \forall x, y$$

but $\underbrace{\subseteq \cdot (\!|b, i|\!)}_{= id_{\mathbb{N}}} \cdot \subseteq = \subseteq \cdot id_P$

hence $\subseteq = \subseteq \iff$ True ✓

□

Injectivity Generalised

$$Z \underset{y}{\overset{x}{\rightrightarrows}} A \xhookrightarrow{f} B$$

monomorphism

$$\boxed{f \circ x = f \cdot y \Rightarrow x = y}$$

In Set

$x \in A$
$y \in A$
and $\boxed{f(x) = f(y) \Rightarrow x = y}$

Example

<u>In Set</u> Define the boolean object

$2 = 1 + 1$

① True = $inl(*)$

False = $inr(*)$

We can define P in two ways

$P : \mathbb{N} \to 2$

$P(n) = even(n) \vee odd(n)$

$(\vee) : 2 \times 2 \to 2$

$P = \{ n \in \mathbb{N} \mid even(n) \vee odd(n) \}$

Define

$$\begin{array}{ccc} 1 & \xrightarrow{b} & P \\ * & \longmapsto & \emptyset \end{array}$$

and

$$\begin{array}{ccc} P & \xrightarrow{i} & P \\ n & \longmapsto & n+1 \end{array}$$

- We should prove b and i are well-defined, that is, that $b(*)$ and $i(n)$ are either even or odd.
- Then we have to prove that $\subseteq$ makes the triangle and the rectangle commute ① ②

Then by previous exercise we completed the proof.