STLC

Simply Typed λ-Calculus

Inductive Set

Functions

$$A, B ::= \text{unit} \mid \text{nat} \mid A \times B \mid A + B \mid A \to B$$

Inductive Set $\quad \mathbb{1} \quad \mathbb{N} \quad$ Products $\quad$ Either (Haskell) Sum

$$t ::= x \mid \underline{n} \mid (t_1, t_2) \mid \text{prj}_1(t) \mid \text{prj}_2(t) \mid \ldots$$

()

$$\mid \text{case } t \text{ of } \{\text{inl}(x) \Rightarrow t_1 \,;\, \text{inr}(y) \Rightarrow t_2\}$$

binder $\quad$ binder

$$\mid \text{inl}(t) \mid \text{inr}(t) \mid \lambda x.t. \mid t_1 t_2$$

$$\mid \text{fold}$$

Type System

A context is a finite list of pairs of the form

$$\Gamma = x_1 : A_1, \ldots, x_n : A_n$$

Some Typing rules

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{}{\Gamma \vdash () : \text{unit}} \qquad \frac{}{\Gamma \vdash \underline{n} : \text{nat}}$$

"identity" $\quad$ True

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B} \qquad \frac{\Gamma \vdash t_1 : A \to B \quad \Gamma \vdash t_2 : A}{\Gamma \vdash t_1 t_2 : B}$$

Denotational Semantics (Categorical)

Take any category $\mathbb{C}$ with objects:

$$1, \mathbb{N}, A \times B, A + B, B^A$$

NN

In general any category with products and exponentials is called <u>cartesian closed</u>.

1) Interpretation of types $[\![\ ]\!] : \text{Types} \to \text{Obj}(\mathbb{C})$

"Types are Objects"

2) Interpretation of programs $[\![\cdot]\!] : \text{Programs} \to \text{Arr}(\mathbb{C})$

"Programs are Arrows"

We only interpret <u>well-typed programs</u>

for $\Gamma \vdash t : A$ $\quad \boxed{[\![\Gamma \vdash t : A]\!] \in \text{Arr}(\mathbb{C})}$ Notation

means "For all Γ, t, A, s.t $\Gamma \vdash t : A$ then $[\![t]\!] \in \text{Arr}(\mathbb{C})$"

3) Interpretation of contexts

Contexts $[\![x_1 : A_1, \dots, x_n : A_n]\!] = [\![A_1]\!] \times \dots \times [\![A_n]\!]$

Types. by induction

$[\![\text{unit}]\!] = 1$

$[\![\text{nat}]\!] = \mathbb{N}$

$[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$

$[\![A + B]\!] = [\![A]\!] + [\![B]\!]$

$[\![A \to B]\!] = [\![B]\!]^{[\![A]\!]}$

Programs by induction on $\Gamma \vdash t : A$

variable case $[\![\Gamma \vdash x : A]\!] : [\![\Gamma]\!] \xrightarrow{\pi_i} [\![A]\!]$ we know $x : A \in \Gamma$

$$(\dots, \underset{\uparrow \text{ ith}}{x}, \dots) \in \underbrace{[\![A_1]\!] \times \dots \times [\![A]\!] \times \dots \times [\![A_n]\!]}_{[\![\Gamma]\!]} \longrightarrow [\![A]\!]$$

Numbers $[\![\Gamma \vdash \underline{n} : \mathbb{N}]\!] : [\![\Gamma]\!] \longrightarrow [\![\text{nat}]\!]$

$$\begin{tikzcd}
{[\![\Gamma]\!]} \arrow[r] \arrow[dr, dotted] & {[\![\text{nat}]\!] = \mathbb{N}} \\
& 1 \arrow[u, "n"']
\end{tikzcd}$$

Products $[\![\Gamma \vdash (t_1, t_2) : A \times B]\!] : [\![\Gamma]\!] \xrightarrow{\langle [\![t_1]\!], [\![t_2]\!] \rangle} [\![A \times B]\!]$

$$\begin{tikzcd}
{[\![A]\!]} & {[\![A]\!] \times [\![B]\!]} \arrow[l] \arrow[r, "\pi_2"] & {[\![B]\!]} \\
& {[\![\Gamma]\!]} \arrow[ul, "{[\![t_1]\!]}"] \arrow[u, dotted, "{\langle [\![t_1]\!], [\![t_2]\!] \rangle}"'] \arrow[ur, "{[\![t_2]\!]}"']
\end{tikzcd}$$

$[\![t_1]\!] : [\![\Gamma]\!] \to [\![A]\!]$

$[\![t_2]\!] : [\![\Gamma]\!] \to [\![B]\!]$

Projections $\quad [\![\Gamma \vdash proj_1(t) : A]\!] : [\![\Gamma]\!] \xrightarrow{\pi_1 \circ [\![t]\!]} [\![A]\!]$

Case Analysis $\quad [\![\Gamma \vdash \text{case } t \text{ of } \{inl(x) \Rightarrow t_1 ; inr(y) \Rightarrow t_2\} : C]\!] : [\![\Gamma]\!] \to [\![C]\!]$

$[\Lambda t_1, \Lambda t_2] \circ [\![t]\!]$

Ingredients $\leftarrow$ things we know by induction

1) $[\![\Gamma \vdash t : A+B]\!] : [\![\Gamma]\!] \longrightarrow [\![A+B]\!]$

2) $[\![\Gamma, x : A \vdash t_1 : C]\!] : [\![\Gamma, x : A]\!] \to [\![C]\!] \quad \Longleftrightarrow \quad \Lambda [\![t_1]\!] : [\![\Gamma]\!] \to$

$[\![\Gamma, x : A]\!] = [\![\Gamma]\!] \times [\![A]\!]$

3) $[\![\Gamma, y : B \vdash t_2 : C]\!] : [\![\Gamma, y : B]\!] \to [\![C]\!]$

$[\![\Gamma, y : B]\!] = [\![\Gamma]\!] \times [\![B]\!]$

$$\begin{tikzcd}
 & {[\![\Gamma]\!]} \arrow[d, "{[\![t]\!]}"] & \\
{[\![A]\!]} \arrow[r, "inl"] \arrow[dr, "\Lambda t_1"'] & {[\![A]\!] + [\![B]\!]} \arrow[d, dashed, "{[\Lambda t_1, \Lambda t_2]}"] & {[\![B]\!]} \arrow[l, "inr"'] \arrow[dl, "\Lambda t_2"] \\
 & {([\![C]\!])^{[\![\Gamma]\!]}} &
\end{tikzcd}$$

λ-abstraction $\quad [\![\Gamma \vdash \lambda x. t : A \to B]\!] : [\![\Gamma]\!] \xrightarrow[\text{curry}(t)]{\Lambda t} [\![A \to B]\!] = [\![B]\!]^{[\![A]\!]}$

$[\![\Gamma, x : A \vdash t : B]\!] : [\![\Gamma]\!] \times [\![A]\!] \longrightarrow [\![B]\!]$

$$\text{curry} : \mathbb{C}(\Gamma \times A, B) \underset{(\vee)}{\overset{(\Lambda)}{\cong}} \mathbb{C}(\Gamma, B^A) : \text{uncurry}$$

Application $[\![\Gamma \vdash t_1 t_2 : B]\!] : [\![\Gamma]\!] \xrightarrow{\epsilon \cdot \langle [\![t_1]\!], [\![t_2]\!] \rangle} [\![B]\!]$ ← Want

$[\![\Gamma \vdash t_1 : A \to B]\!] : [\![\Gamma]\!] \to [\![B]\!]^{-}$
$[\![\Gamma \vdash t_2 : A]\!] : [\![\Gamma]\!] \to [\![A]\!]$ } Ind. hypothesis

$$\begin{tikzcd}
{[\![B]\!]^{[\![A]\!]} \times [\![A]\!]} \arrow[r, "\epsilon"] & {[\![B]\!]} \\
{[\![\Gamma]\!] \times [\![A]\!]} \arrow[u, "{[\![t_1]\!] \times [\![A]\!]}"] & \\
{[\![\Gamma]\!]} \arrow[u, "{\langle id_\Gamma, [\![t_2]\!] \rangle}"] \arrow[uu, bend right=60, "{\langle [\![t_1]\!], [\![t_2]\!] \rangle}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B^A \times A \arrow[r, "\epsilon"] & B \\
\Gamma \times A \arrow[u, "\Lambda f \times A"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Some Properties

Lemma (Substitution)

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash u : A}{\Gamma \vdash [u/x]t : B}$$

$$[\![\Gamma]\!] \xrightarrow{\langle id, [\![u]\!] \rangle} [\![\Gamma]\!] \times [\![A]\!] \xrightarrow{[\![t]\!]} [\![B]\!]$$

Substitution in λ is composition of arrows

Lemma (Weakening)

$\Gamma \vdash s : C$ then so is $\Gamma, x : A \vdash s : C$ then ⟹

$$\begin{tikzcd}
{[\![\Gamma \times A]\!]} \arrow[r, "{[\![s]\!]}"] \arrow[d, "\pi_2"] & {[\![C]\!]} \\
{[\![\Gamma]\!]} \arrow[ur, "{[\![s]\!]}"'] &
\end{tikzcd}$$

Lemma (β, η rules)

$$(\lambda x. t) u \approx_\beta [u/x]t$$

$$[\![(\lambda x. t) u]\!] \doteq [\![[u/x]t]\!]$$

$$\lambda x. (s x) \approx_\eta s$$

$$[\![\lambda x. s x]\!] = [\![s]\!]$$

On Computational Adequacy and Full Abstraction

Soundness

operational semantics

$$t \Downarrow v \implies [\![t]\!] = [\![v]\!]$$

model

"the model is agnostic to step reductions"

Completeness

$$[\![t]\!] = [\![v]\!] \implies t \Downarrow v$$

We can use the model to reason about operational semantics

Contextual Equivalence

$$t \approx s \triangleq \forall \mathbb{C}.\ \mathbb{C}[t] \Downarrow v \iff \mathbb{C}[s] \Downarrow v$$

Full Abstraction

$$\boxed{[\![t]\!] = [\![s]\!] \iff s \approx t}$$

Computational Adequacy

$$= \forall \mathbb{C}.\ \mathbb{C}[t] \Downarrow v \iff \mathbb{C}[s] \Downarrow v$$

Computational Adequacy follows from Soundness + Completeness + Compositionality

Proof.

Assume $[\![t]\!] = [\![s]\!]$

Let $\mathbb{C} \in$ context

Assume $\mathbb{C}[t] \Downarrow v$

by Soundness $[\![\mathbb{C}[t]]\!] = [\![v]\!]$

by Soundness $[\![C[t]]\!] = [\![v]\!]$

because $[\![\ ]\!]$ is defined by induction

$$[\![C[t]]\!] = [\![C]\!] \circ [\![t]\!] = [\![v]\!]$$

$$\Rightarrow [\![C]\!] \circ [\![s]\!] = [\![v]\!]$$

QED

$$\Rightarrow [\![C[s]]\!] = [\![v]\!] \xrightarrow{\text{Completeness}} C[s] \Downarrow v \quad \square$$