Functors and Nat. Transformations

$F : \mathbb{C} \longrightarrow \mathbb{D}$

Object part and a arrow part (functorial action)

For $X \in \mathrm{Obj}(\mathbb{C})$, FX $\qquad$ $F(f)$, for $f : A \to B$

$$\varphi : F \xrightarrow{\cdot} G$$

$$\varphi_X : FX \longrightarrow GX \quad \text{in } \mathbb{D}$$

s.t.

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} FX \arrow[r, "\varphi_X"] \arrow[d, "Ff"'] & GX \arrow[d, "G(f)"] \\ FY \arrow[r, "\varphi_Y"] & GY \end{tikzcd}$$

in $\mathbb{C}$

$R \subseteq A \times B$

$$\begin{tikzcd} FX \arrow[r, "\varphi_X"] \arrow[d, "FR"'] & GX \arrow[d, "GR"] \\ FY \arrow[r, "\varphi_X"] & GY \end{tikzcd}$$

$f_x \quad :: \quad \underline{\text{Maybe}}\ x \longrightarrow \underline{\text{List}}\ x$

Let $\mathbb{C}$, $\mathbb{D}$, $\underline{\mathbb{C} \times \mathbb{D}}$ is the <u>product category</u>

$\mathbb{C} \times \mathbb{D}$ objects are pairs (X, Y)

s.t $X \in \mathbb{C}$, $Y \in \mathbb{D}$

$$f : (X, Y) \longrightarrow (X', Y')$$

$$f_1 : X \to X' \text{ in } \mathbb{C}$$

$$f_2 : Y \to Y' \text{ in } \mathbb{D}$$

Exercise that there exists an identity morphism and morphism in $\mathbb{C} \times \mathbb{D}$ compose

Let $\mathbb{C}, \mathbb{D}$ be categories.

$[\mathbb{C}, \mathbb{D}]$ <u>functor category</u>

objects are <u>functors</u> $F : \mathbb{C} \to \mathbb{D}$

arrows are <u>natural transformations</u>

$$F \stackrel{\cdot}{\to} G$$

The category of categories called CAT

with objects are categories $\mathbb{C}, \mathbb{D}$

what are the morphisms? <u>Functors</u>

$$\mathbb{C} \longrightarrow \mathbb{D}$$

The Opposite Category

For a category $\mathbb{C}$, we call $\mathbb{C}^{op}$ the category

where objects are

$$Obj(\mathbb{C}^{op}) = Obj(\mathbb{C})$$

the arrows are reversed meaning

$$\mathbb{C}^{op}(A,B) \triangleq \mathbb{C}(B,A)$$

$$hom_{\mathbb{C}^{op}}(A,B) = hom_{\mathbb{C}}(B,A)$$

$$f^{op} \in \mathbb{C}^{op}(A,B) \iff f \in \mathbb{C}(B,A)$$

$$F: \mathbb{C}(A,B) \longrightarrow \mathbb{D}(FA,FB)$$

$$(-)^X : \mathbb{C}(A,B) \longrightarrow \mathbb{C}(A^X, B^X)$$

$$(f)^X = curry(f \cdot \in)$$

$$\left(\begin{tikzcd} A^X \times X \arrow[r] \arrow[dr, "\in"'] & B \\ & A \arrow[u, "f"'] \end{tikzcd}\right) \xrightarrow{curry} A^X \to B^X$$

in Set $\quad f^X = \lambda g : X \to A.\ \lambda x. f \cdot g(x)$

A <u>functorial action</u> has type:

$$F : \mathbb{C}(A,B) \to \mathbb{C}(FA, FB)$$

Homset extends to a <u>covariant functor</u>

$f : A \to B \qquad \mathbb{C}(X,-) : \mathbb{C} \to Set$ Covariant

$$\underbrace{\mathbb{C}(X,f)}_{Ff=} : \underbrace{\mathbb{C}(X,A)}_{FA} \longrightarrow \underbrace{\mathbb{C}(X,B)}_{FB''} \quad \text{in } \underline{Set}$$

The homset extends to a <u>contravariant functor</u>

$$\underbrace{\mathbb{C}(-,X)}_{\text{the set of morphisms}} : \mathbb{C}^{op} \to Set$$

(if $\mathbb{C}$ is locally small)

Contravariant

$$\overset{op}{f} : A \to B \qquad \underbrace{\mathbb{C}(f,X)}_{Ff} : \underbrace{\mathbb{C}(A,X)}_{FA} \to \underbrace{\mathbb{C}(B,X)}_{FB}$$

this lives in $\underline{C^{op}}$!! $\longrightarrow$ otherwise you can't define the functor

<u>Example</u> (of Duality in Category theory)

$\mathbb{C}$ has products $\iff$ $\mathbb{C}^{op}$ has coproducts

$$\begin{tikzcd} X & X \times Y \arrow[l] \arrow[r] & Y \\ & Z \arrow[ul] \arrow[ur] \arrow[u, dashed] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[dr] & X+Y \arrow[d, dashed] & Y \arrow[l] \arrow[dl] \\ & Z & \end{tikzcd}$$