

# Functors and Nat. Transformations

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

Object part and a arrow part (functorial action)

For  $X \in \text{Obj}(\mathcal{C})$ ,  $FX$

$F(f)$ , for  $f: A \rightarrow B$

$$\varphi: F \longrightarrow G$$

$$\varphi_x: FX \longrightarrow GX \quad \text{in } \mathcal{D}$$

s.t.

$$\begin{array}{ccccc} A & & FX & \xrightarrow{\varphi_x} & GX \\ f \downarrow & & f \downarrow & & \downarrow G(f) \\ B & & FY & \xrightarrow{\varphi_y} & GY \end{array}$$

in  $\mathcal{C}$

$$R \subseteq A \times B$$

$$\begin{array}{ccc} \boxed{\begin{array}{c} FX \\ \downarrow FR \\ FY \end{array}} & \xrightarrow{\varphi_x} & \boxed{\begin{array}{c} GX \\ \downarrow GR \\ GY \end{array}} \end{array}$$

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$$f\ x \quad :: \quad \underline{\text{Maybe}}\ x \rightarrow \underline{\text{List}}\ x$$


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Let  $\mathcal{C}, \mathcal{D}$ ,  $\mathcal{C} \times \mathcal{D}$  is the product category

$\mathcal{C} \times \mathcal{D}$  objects are pairs  $(X, Y)$

s.t  $X \in \mathcal{C}, Y \in \mathcal{D}$

$$f: (X, Y) \longrightarrow (X', Y')$$

$$f_1: X \longrightarrow X' \text{ in } \mathcal{C}$$

$$f_2: Y \longrightarrow Y' \text{ in } \mathcal{D}$$

Exercise that there exists an identity morphism and morphism in  $\mathcal{C} \times \mathcal{D}$  compose

Let  $\mathcal{C}, \mathcal{D}$  be categories.

$[\mathcal{C}, \mathcal{D}]$  functor category

objects are functors  $F: \mathcal{C} \rightarrow \mathcal{D}$

arrows are natural transformations

$$F \xrightarrow{\quad} G$$

The category of categories called CAT

with objects are categories  $\mathcal{C}, \mathcal{D}$

what are the morphisms? Functors

$$\mathcal{C} \longrightarrow \mathcal{D}$$



## The Opposite Category

for a category  $\mathcal{C}$ , we call  $\mathcal{C}^{\text{op}}$  the category

whose objects are

$$\text{Obj}(\mathcal{C}^{\text{op}}) = \text{Obj}(\mathcal{C})$$

the arrows are reversed meaning

$$\mathcal{C}^{\text{op}}(A, B) \triangleq \mathcal{C}(B, A)$$

$$\text{hom}_{\mathcal{C}^{\text{op}}}(A, B) = \text{hom}_{\mathcal{C}}(B, A)$$

$$f^{\text{op}} \in \mathcal{C}^{\text{op}}(A, B) \iff f \in \mathcal{C}(B, A)$$

$$F: \mathcal{C}(A, B) \longrightarrow \mathcal{D}(FA, FB)$$

$$(-)^{\times}: \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A^{\times}, B^{\times})$$

$$(f)^{\times} = \underline{\text{corry}}(f \cdot e)$$

$$\left( \begin{array}{ccc} A^{\times} \times X & \rightarrow & B \\ e \searrow & \nearrow f & \\ & A & \end{array} \right) \xrightarrow{\text{corry}} A^{\times} \rightarrow B^{\times}$$

in  $\text{Set}$   $f^{\times} = \lambda g: X \rightarrow A. \lambda x. f \cdot g(x)$

A functorial action has type:

$$F: \mathcal{C}(A, B) \rightarrow \mathcal{C}(FA, FB)$$

Homset extends to a covariant functor

$$f: A \rightarrow B \quad \mathcal{C}(X, -): \mathcal{C} \rightarrow \text{Set} \quad \text{Covariant}$$

$$\underbrace{\mathcal{C}(X, f)}_{\neq f} : \underbrace{\mathcal{C}(X, A)}_{\neq A} \rightarrow \underbrace{\mathcal{C}(X, B)}_{\neq B} \quad \text{in Set}$$

The homset extends to a contravariant functor

$$\underbrace{\mathcal{C}(-, x)}_{\text{the set of morphisms}} : \mathcal{C}^{\text{op}} \rightarrow \text{Set} \quad (\text{if } \mathcal{C} \text{ is locally small})$$

Contravariant

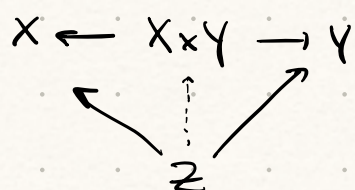
$$f: A \rightarrow B \quad \mathcal{C}(f, x): \mathcal{C}(A, x) \rightarrow \mathcal{C}(B, x)$$

$\neq f \quad \neq A \quad \neq B$

this lives in  $\mathcal{C}^{\text{op}}$ !!  $\rightarrow$  otherwise you can't define the functor

Example (of Dualities in Category theory)

$\mathcal{C}$  has products  $\iff$



$\mathcal{C}^{\text{op}}$  has coproducts

