# The Yoneda Lemma

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1. Recap
   A. Functors
   B. Natural Transformations
2. Representable Functor
3. The Mini Yoneda Lemma for Type Theorists
4. The Yoneda Lemma

$$F : \mathbb{C} \longrightarrow \mathbb{D}$$

$$F : Obj(\mathbb{C}) \longrightarrow Obj(\mathbb{D}) \qquad X \in \mathbb{C} \qquad FX$$

$$F : Arr(\mathbb{C}) \longrightarrow Arr(\mathbb{D}) \qquad f : \mathbb{C} \qquad Ff$$

$$F : \mathbb{C}(A, B) \longrightarrow \mathbb{D}(FA, FB)$$

$$\varphi : F \stackrel{\cdot}{\longrightarrow} G$$

$$X \in \mathbb{C} \qquad \varphi_X : FX \longrightarrow GX$$

$$
\begin{tikzcd}
FX \arrow[r, "\varphi_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "\varphi_Y"'] & GY
\end{tikzcd}
$$

$$\varphi_{\mathbb{N}} : \text{List } \underline{\mathbb{N}} \longrightarrow \text{Maybe } \underline{\mathbb{N}}$$

$$\varphi_{\mathbb{N}}\ [\,] = \text{Nothing}$$

$$\varphi_{\mathbb{N}}\ (x:xs) = \text{Just } (x+1)$$

$$\varphi_{\mathbb{B}} : \text{List } \mathbb{B} \longrightarrow \text{Maybe } \mathbb{B}$$

$$\varphi_{\mathbb{B}}\ [\,] = \text{Nothing}$$

$$\varphi_{\mathbb{B}}\ (x:xs) = \text{Just } (\text{flip } x)$$

$$
\begin{tikzcd}
(n:ns) \arrow[rr, bend left] \arrow[dd, bend right] & & \text{Just } (n+1) \arrow[dd, bend left] \\
& \text{List } \mathbb{N} \arrow[r, "\varphi_{\mathbb{N}}"] \arrow[d, "\text{List } f"] & \text{Maybe } \mathbb{N} \arrow[d, "\text{Maybe } f"] \\
& \text{List } \mathbb{B} \arrow[r, "\varphi_{\mathbb{B}}"'] & \text{Maybe } \mathbb{B} \\
f(n):(\text{List } f)ns \arrow[rr, bend right] & & \text{Just } (\text{flip}(f(n)))
\end{tikzcd}
$$

$$\mathbb{N} \xrightarrow{f} \mathbb{B}$$

$$\text{Just } (f(n+1)) = \text{Just } (\text{flip}(f(n)))$$

$$f(n+1) = \mathrm{flip}(f(n))$$

$$f : \mathbb{N} \to \mathbb{B}$$
$$f\ n = \mathrm{True}$$

Representable

Homset extends to a functor

$$\mathbb{C}(-,-) : \mathbb{C}^{op} \times \mathbb{C} \to \mathrm{Set}$$

A functor $F : \mathbb{C} \to \mathrm{Set}$ is representable whenever $\exists\ A \in \mathrm{Obj}(\mathbb{C})$ s.t

$$F \cong \mathbb{C}(A, -)$$

$$\mathrm{List} \cong \mathrm{Set}(\mathbb{N}, -)$$

is it true that for all $A$ $\mathrm{List}\,A \cong \mathrm{Set}(\mathbb{N}, A)$

$$\mathrm{List}\,A \not\cong \mathbb{N} \to A$$

In Set

$$Str\ A \cong \mathbb{N} \to A$$

$$xs \mapsto \lambda n.\ head\ (drop\ n\ xs)$$

$$f \cdot succ \mapsfrom f$$

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(Mini) Yoneda Lemma for Type Theorists

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash R(t) : B}$$

Task is find and interpretation for R given a morphism $A \xrightarrow{f} B$

$$\in \mathbb{C}(\Gamma, B)$$

$$[\![\Gamma \vdash R(t) : B]\!] : [\![\Gamma]\!] \to [\![B]\!]$$

HpInd

$$[\![\Gamma]\!] \to [\![A]\!]$$

$$\frac{\Gamma \xrightarrow{t} A}{\Gamma \xrightarrow{R(t)} B} \qquad A \xrightarrow{f} B$$

Mini Yoneda $\quad \mathbb{C}$ locally small

$$\mathbb{C}(A,B) \cong \mathbb{C}(-,A) \xrightarrow{\cdot} \mathbb{C}(-,B)$$

p6

$$\cong \boxed{\forall \Gamma.\ \boxed{\mathbb{C}(\Gamma, A)} \to \boxed{\mathbb{C}(\Gamma, B)}}$$

$$\frac{[\![\Gamma \vdash t : A]\!] \in \boxed{\mathbb{C}(\Gamma, A)}}{[\![\Gamma \vdash R(t) : B]\!] \in \boxed{\mathbb{C}(\Gamma, B)}}$$

$$\mathbb{C}(A, B) \underset{\psi}{\overset{\varphi}{\cong}} \mathbb{C}(-, A) \xrightarrow{\bullet} \mathbb{C}(-, B)$$

$$\varphi(f)(g : \Gamma \to A) = \underbrace{\lambda\gamma.\ f(g(\gamma))}_{\Gamma \to B}$$

$$= f \cdot g$$

$$\psi(\alpha) = \alpha_A(\mathrm{id}_A)$$

$$\mathbb{C}(A, A) \to \mathbb{C}(A, B)$$

$$\frac{\Gamma \vdash t : \boxed{A}}{\Gamma \vdash R(t) : \boxed{B}} \quad f$$

Yoneda $\quad F : \mathbb{C}^{op} \longrightarrow \mathbf{Set}$

$$FC \cong \mathbb{C}(-,C) \xrightarrow{\cdot} F$$

for example $F = \mathbb{C}(-,B)$

$$\underline{FA} \cong \mathbb{C}(-,A) \xrightarrow{\cdot} \mathbb{C}(-,B)$$

$\mathbb{C}(A,B)$