Yoneda

let $F : \mathbb{C}^{op} \to Set$ (contravariant set-valued functor)

$\mathbb{C}$ ← locally small $\mathbb{C}(X,Y)$ is a set for all X, Y.

$$FC \cong \underbrace{\mathbb{C}(C,-) \xrightarrow{\cdot} F}_{\text{Nat transf.}}$$

Mini Yoneda

$$F = \mathbb{C}(-,B) \Rightarrow \mathbb{C}(A,B) \cong \mathbb{C}(-,A) \xrightarrow{\cdot} \mathbb{C}(-,B)$$
$$= Nat(\mathbb{C}(-,A), \mathbb{C}(-,B))$$

For every F functor preserves isomorphisms

$$X \underset{g}{\overset{f}{\cong}} Y \Rightarrow FX \cong FY$$

$\Leftarrow$?

1) $g \circ f = id_X$ $\quad F(f \circ g) = F\, id_X \Rightarrow Ff \circ Fg = id_{FX}$

2) $f \circ g = id_Y$

FULL
When a functor is surjective on morphisms

FAITHFUL
When it is injective on morphisms

if F is Fully Faithfull then

$$\boxed{X \cong Y \iff FX \cong FY}$$

$$\begin{array}{ccccc} & \exists f' & Ff' = f & & f' \\ FX \underset{g}{\overset{f}{\cong}} FY & \text{surjectivity} & & \Rightarrow & X \underset{g'}{\overset{f'}{\cong}} Y \\ & \exists g' & Fg' = g & & \end{array}$$

$Ff' = f$

$Fg' = g$

$$\boxed{g' \circ f' = id_X}$$

$Fz = id_X \Rightarrow z = id$ (injectivity)

$$F : C(X,Y) \longrightarrow C(FX,FY)$$

$$Fg' \circ Ff' = id_{FX} = F id_X$$

$$\|$$

$$F(g' \circ f') = F id_X \Rightarrow g' \circ f' = id_X \qquad \square$$

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2 Yoneda Embedding

$$y = C(-,=) : C \longrightarrow Set^{C^{op}}$$ Fully Faithfull

$$A^{B^C} \cong A^{B \times C} \Longleftarrow y(A^{B^C}) \cong y(A^{B\times C})$$

uncurry

$$C(X, \boxed{A^{B^C}}) \cong C(X \times C, A^B)$$

uncurry

$$\cong C((X \times C) \times B, A)$$

$$\cong \mathbb{C}(X \times (C \times B), A)$$

$$\cong \mathbb{C}(X \times (B \times C), A)$$

currying

$$\cong \mathbb{C}(X, \boxed{A^{B \times C}})$$

$$(A+B) \times C \cong (A \times C) + (B \times C)$$

$$y((A+B) \times C) = \mathbb{C}((A+B) \times C, X)$$ curry

$$\cong \mathbb{C}(A+B, X^C)$$ coproduct

$$\cong \mathbb{C}(A, X^C) \times \mathbb{C}(B, X^C)$$ uncurry

$$\cong \mathbb{C}(A \times C, X) \times \mathbb{C}(B \times C, X)$$

$$\cong \mathbb{C}(A \times C + B \times C, X) = y((A \times C) + (B \times C))$$

$\square$