Groups, Categories and the Yoneda Lemma

A **GROUP** G is a set X together with

1) a unit $1_G \in X$ element
2) a multiplication

$$(\cdot) : G \times G \longrightarrow G$$

3) $(-)^{-1} : G \longrightarrow G$

subject to equations

$$1_G \cdot x = x \cdot 1 = x \qquad \text{unitary laws}$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \qquad \text{associativity}$$

$$x \cdot x^{-1} = 1_G \qquad \text{inverse}$$

Examples

$(\mathbb{Z}, +, 0)$ set of integers with zero and addition

A group homomorphism between two groups $(G, \cdot, 1_G)$ and $(H, +, 1_H)$ is a function $f : G \longrightarrow H$ st

$$f(x \cdot y) = f(x) + f(y)$$

$$f(1_G) = 1_H$$

The <u>category of Groups</u> has groups as objects and group homomorphisms as morphisms, is denoted by Grp.

A <u>group</u> $G = (|G|, \cdot, 1_G)$ <u>is a category</u> with one object $*$ and where morphisms are the elements of the group.

$$\begin{tikzcd} * \arrow[loop left, "1_G"] \arrow[loop below, "x"] \arrow[loop right, "y"] \end{tikzcd}$$

the <u>identity morphism</u> is the unit of the group and <u>composition</u> is the multiplication of the group.

the homset is the set of all elements of G hence $G(*,*) = |G|$.

The homset functor

$$G(*,-) : G \longrightarrow \mathrm{Set}$$

is defined

$$G(*,f) : G(*,*) \longrightarrow G(*,*)$$
$$G(*,f)(g) = f \cdot g$$

Given a group G, $\mathrm{Sym}(G)$ is the group of bijective homomorphisms

$$(G \longrightarrow G, \circ, id_G)$$

together with function composition and the identity map

Celey's Theorem

Every group G is isomorphic to a subgroup N of $Sym(G)$ where $Sym(G)$ is the symmetry group, the group of bijective homomorphisms together with composition and the identity function

$$\varphi : \mathbb{Z} \longrightarrow Sym(\mathbb{Z}) : \varphi$$

trivially a bijection

$$x \longmapsto \boxed{\lambda y.\ x+y}$$

$$(\lambda z.\ z-x) \circ (\lambda y.\ x+y) = id$$

and viceversa

$$g(0) \longleftarrow g$$

Prove $\mathbb{Z} \cong \varphi(\mathbb{Z})$.

$$\forall x \in \mathbb{Z} \quad \varphi(\varphi(x)) = \varphi(\lambda y.\ x+y) = x+0$$
$$= x$$

$$\forall g \in \varphi(\mathbb{Z}). \quad g = \lambda y.\ x+y.$$
$$\varphi(\varphi(g)) = \varphi(g(0)) = \varphi(x) = g$$

Caley's theorem is __JUST__ the Yoneda Lemma

Yoneda Lemma

$$FC \cong \mathbb{C}(C,-) \overset{\cdot}{\longrightarrow} F$$

__Proof.__

$$\varphi(x)(g : C \to X) = F(g)(x)$$

$$\varphi(\alpha) = \alpha(id_C)$$

For $F = \mathbb{C}(A,-)$

$$\mathbb{C}(A,C) \cong \mathbb{C}(C,-) \overset{\cdot}{\longrightarrow} \mathbb{C}(A,-)$$

Take $F : G \to Set$ to be the homset functor

$$G(*,-) : G \to Set$$

where G is a group viewed as a category.

Then

$$\underbrace{G(*,*)}_{G} \cong G(*,-) \xrightarrow{\cdot} G(*,-)$$

$$G \cong G \xrightarrow{\cdot} G$$

$$G(*,-) : G \longrightarrow \mathbf{Set}$$