

# Groups, Categories and the Yoneda Lemma

A GROUP  $G$  is a set  $X$  together with

1) a unit  $1_G \in X$  element

2) a multiplication

$$(\cdot): G \times G \longrightarrow G$$

3)  $(-)^{-1}$ :  $G \longrightarrow G$

subject to equations

$$1_G \cdot x = x \cdot 1 = x \quad \text{unitary laws}$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \text{associativity}$$

$$x \cdot x^{-1} = 1_G \quad \text{inverse}$$

## Examples

$(\mathbb{Z}, +, 0)$  set of integers with zero and addition

A group homomorphism between two groups  $(G, \cdot, 1_G)$  and  $(H, +, 1_H)$

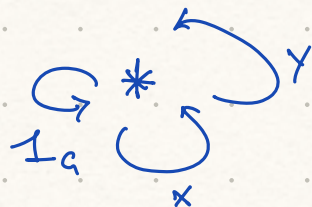
is a function  $f: G \longrightarrow H$  st

$$f(x \cdot y) = f(x) + f(y)$$

$$f(1_G) = 1_H$$

The Category of Groups has groups as objects and group homomorphisms as morphisms, is denoted by  $\text{Grp}$ .

A group  $G = (|G|, \cdot, 1_G)$  is a category with one object  $*$  and where morphisms are the elements of the group.



the identity morphism is the unit of the group and composition is the multiplication of the group.

the homset is the set of all elements of  $G$  hence  $G(*, *) = |G|$ .

The homset functor

$$G(*, -) : G \rightarrow \text{Set}$$

is defined

$$G(*, f) : G(*, *) \rightarrow G(*, *)$$

$$G(*, f)(g) = f \cdot g$$

Given a group  $G$ ,  $\text{Sym}(G)$  is the group of bijective homomorphisms

$$(G \rightarrow G, \circ, \text{id}_G)$$

together with function composition and the identity map



# Caley's Theorem

Every group  $G$  is isomorphic to a subgroup  $N$  of  $\text{Sym}(G)$  where  $\text{Sym}(G)$  is the symmetry group, the group of bijective homomorphisms together with composition and the identity function

$$\begin{aligned} \varphi: \mathbb{Z} &\longrightarrow \text{Sym}(\mathbb{Z}) : \varphi \\ x &\longmapsto \boxed{\lambda y. x+y} \end{aligned} \quad \begin{aligned} &\text{trivially a bijection} \\ &(\lambda z. z-x) \circ (\lambda y. x+y) = \text{id} \\ &\text{and viceversa} \end{aligned}$$
$$g(0) \longleftarrow g$$

Prove  $\mathbb{Z} \cong \varphi(\mathbb{Z})$ .

$$\forall x \in \mathbb{Z} \quad \varphi(\varphi(x)) = \varphi(\lambda y. x+y) = x+0 = x$$

$$\forall g \in \varphi(\mathbb{Z}). \quad g = \lambda y. x+y.$$

$$\varphi(\varphi(g)) = \varphi(g(0)) = \varphi(x) = g$$

Coley's theorem is Just the  
Yoneda Lemma

Yoneda Lemma

$$FC \cong C(C, -) \longrightarrow F$$

Proof.

$$\varphi(x)(g: C \rightarrow X) = F(g)(x)$$

$$\varphi(\alpha) = \alpha(id_C)$$

For  $F = C(A, -)$

$$C(A, C) \cong C(C, -) \longrightarrow C(A, -)$$

Take  $F: G \rightarrow \text{Set}$  to be  
the homset functor

$$G(*, -): G \rightarrow \text{Set}$$

where  $G$  is a group viewed as  
a category.



Then

$$G(*, *) \cong G(*, -) \longrightarrow G(*, -)$$

$$G \cong G \longrightarrow G$$

$$G(*, -) : G \longrightarrow \mathbf{Set}$$

