A Monad is Just (!) a Monoid in the Category of EndoFunctors

Definition (Monad)

Endofunctor $T: \mathbb{C} \to \mathbb{C}$ - (List: Type → Type)

Unit: $\eta_A : A \longrightarrow TA$ - (return, singleton list)

Multiplication: $\mu_A : TTA \longrightarrow TA$ - "concat"

Concatenation of Lists $\Longrightarrow$ concat $[e_1, e_2, \dots e_n] = e_1 \mathbin{+\!\!+} e_2 \mathbin{+\!\!+} \dots \mathbin{+\!\!+} e_n$

flatMap return

Definition (Monoid)

M is a set

$1_M \in M$

$(+): M \times M \longrightarrow M$

Subject to Equations

$1 + m = m = m + 1$ unitary laws

$(m+n)+r = m+(n+r)$ associativity

Examples

$(\mathbb{N}, +, 0)$ $(\mathbb{Z}, +, 0)$

$\cancel{(\mathbb{N}, -,)}$ $(\mathbb{Z}, *, 1)$

$(\mathbb{N}, *, 1)$

Endofunctor category

$([\mathbb{C},\mathbb{C}], \circ, \mathrm{Id})$

$(\circ) : [\mathbb{C},\mathbb{C}] \times [\mathbb{C},\mathbb{C}] \to [\mathbb{C},\mathbb{C}]$

$T \circ \mathrm{Id} = T$ $\quad \eta \circ T \quad T \circ T \quad T \circ \eta \quad \mathrm{Id} \circ T = T$

$$\begin{tikzcd}
TA \arrow[r, "\eta_{TA}"] \arrow[dr, "="'] & TTA \arrow[d, "\mu"] & TA \arrow[l, "T\eta"'] \arrow[dl, "="] \\
 & TA &
\end{tikzcd}$$

T

$T \circ T \circ T \qquad T \circ \mu \qquad T \circ T$

$$\begin{tikzcd}
TTTA \arrow[r, "T\mu"] \arrow[d, "\mu_{TA}"'] & TTA \arrow[d, "\mu"] \\
TTA \arrow[r, "\mu"'] & TA
\end{tikzcd}$$

$\mu \circ T \qquad T \circ T = \qquad = T$

Monoid in a category

$(\mathbb{C}, \times, 1)$

it is an object M

$1 \xrightarrow{e} M$ unit

$M \times M \xrightarrow{m} M$ multiplication

$(e, x) \quad (x, e) \quad x \quad x$

$$\begin{tikzcd}
M \cong M \times 1 \arrow[r, "{\langle id_M, e \rangle}"] \arrow[dr, "="'] & M \times M \arrow[d, "m"] & M \arrow[l, "{\langle e, id_M \rangle}"'] \arrow[dl, "="] \\
 & M &
\end{tikzcd}$$

$(x, y, z) \qquad (x, yz)$

$$\begin{tikzcd}
M \times (M \times M) \arrow[r, "M \times m"] \arrow[d, "m \times M"'] & M \times M \arrow[d] \\
M \times M \arrow[r] & M
\end{tikzcd}$$

$(xy, z) \qquad xyz$

Happy Summer!