A Memorandum on Kan Extensions and Monads

Marco Paviotti

Abstract

These notes are meant to remind myself of some facts about Kan extension and monads. The first part is devoted to basic definitions about adjunctions, ends and coends, which are needed to explain the proofs later on. The second part is on monads and Kan extensions.

Contents

1 Adjunctions and Monads

Given two functors $L : \mathcal{D} \to \mathcal{C}$ and $R : \mathcal{C} \to \mathcal{D}$ and *adjunction* is an isomorphism of homsets

$$\lfloor \cdot \rfloor : \mathcal{C}(LA, B) \cong \mathcal{D}(A, RB) : \lceil \cdot \rceil$$

which is furthermore natural in A and B. Here $\lfloor \dot{} \rfloor$ and $\lceil \dot{} \rceil$ are the functions witnessing the isomorphism. The adjunction is usually depicted as follows

$$\mathcal{C} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{D}$$

We say that that L is left adjoint and R is right adjoint and it is indicated by $L \vdash R$. As a consequence of the isomorphism, for $f : LA \to B$ and $g : A \to RB$ we have that

$$\lfloor f \rfloor = g \iff f = \lceil g \rceil$$

and because the isomorphism is natural we can derive the *fusion laws*. For $a : A' \to A$, $b : B \to B'$, $f : LA \to B$ and $g : A \to RB$

$$R(b) \cdot \lfloor f \rfloor = \lfloor b \cdot f \rfloor$$
$$\lfloor f \rfloor \cdot a = \lfloor f \cdot L(a) \rfloor$$
$$b \cdot \lceil g \rceil = \lceil R(b) \cdot g \rceil$$
$$\lceil g \rceil \cdot L(a) = \lceil g \cdot a \rceil$$

We can also compute the fusion laws this way.

$$R(b) \cdot \lfloor f \rfloor \cdot a = \lfloor b \cdot f \cdot L(a) \rfloor$$
$$b \cdot \lceil g \rceil \cdot L(a) = \lceil R(b) \cdot g \cdot a \rceil$$

This is really all about adjunctions. All the other definitions and constructions are equivalent to this one. Furthermore, this material is very well covered elsewhere [2, 5, 3] so I will not be covering it further.

What is important for the sake of this notes is that an adjunction gives rise to a monad and a comonad where RL is the monad and LR is the comonad. The unit and counit of the adjunction are defined as follows

$$\eta_A = \lfloor id_{LA} \rfloor$$
$$\epsilon_B = \lceil id_{RB} \rceil$$

and are respectively the unit of the monad and the counit of the comonad generated by the adjunction. The join of the monad $\mu : RLRL \to RL$ is defined as $\mu = R\epsilon_L$ and the cojoin $\delta : LR \to LRLR$ is defined as $\delta = L\eta_R$.

2 Limits and Colimits

Given two objects X and Y in a category, $X \times Y$ forms the product of X and Y. We can generalise this further. Given a functor $D : \mathcal{I} \to \mathcal{C}$ the limit $\varprojlim D$ is an universal object such that for every $I \in \mathcal{I}$, there exists a projection map $\varprojlim DI \xrightarrow{\pi_I} DI$ such that for every morphism $DI_1 \xrightarrow{f} DI_2$ we have

$$\pi_{I_2} = f \cdot \pi_{I_1}$$

and furthermore any other object such as this has a unique morphism into the limit commuting with the projections [2, 5].

The limit is right adjoint to the diagonal functor mapping every object to the constant functor and the colimit is the left adjoint to the diagonal functor.

$$\mathcal{C} \underset{\Delta}{\overset{\varinjlim}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{C}^{\mathcal{I}} \underset{\varprojlim}{\overset{\Delta}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{C}$$

We right down the isomorphisms

$$\mathcal{C}(\varinjlim_{I \in \mathcal{I}} A(I), B) \cong \mathcal{C}^{\mathcal{I}}(A, \Delta B)$$

$$\mathcal{C}^{\mathcal{I}}(\Delta A, B) \cong \mathcal{C}(A, \varprojlim_{I \in \mathcal{I}} B)$$

2.1 Preservation and Creation of (Co)Limits

A functor $H : \mathcal{C} \to \mathcal{D}$ is said to *preserve* limits if, given a diagram $F : \mathcal{I} \to \mathcal{C}$

$$H \varprojlim_{I \in \mathcal{I}} FI \cong \varprojlim_{I \in \mathcal{I}} HFI$$

In other words, H preserves limits if the limit of the diagram obtained by composition with H, namely HF, corresponds with the limit of F applied to H. In particular, such a functor preserves small limits as well. A functor that preserves small (co)limits is called *(co)continuous*.

As a prominent example, the covariant homset functor $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$ preserve limits

$$\mathcal{C}(C, \varprojlim_{I \in \mathcal{I}} FI) \cong \varprojlim_{I \in \mathcal{I}} \mathcal{C}(C, FI) \tag{1}$$

On the other hand, the contravariant homset functor, which may be written as $\mathcal{C}(-, C) = \mathcal{C}^{\mathrm{op}}(C, -) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ carries colimits over to limits in the following sense

$$\mathcal{C}(\varinjlim_{I \in \mathcal{I}} FI, C) \cong \varprojlim_{I \in \mathcal{I}} \mathcal{C}(FI, C) \tag{2}$$

2.2 Dependent product and sum

Let us consider the set $\mathcal{I}$ (or the discrete category $\mathcal{I}$ with only identities). Then the right adjoint to the diagonal functor is called the *dependent product* $\Pi_{I\in\mathcal{I}}B(I)$ for some functor $B : \mathcal{I} \to \mathcal{C}$ and the left adjoint is called the *dependent sum* $\Sigma_{I\in\mathcal{I}}A(I)$ for some functor $A : \mathcal{I} \to \mathcal{C}$.

$$\mathcal{C} \underset{\Delta}{\overset{\Sigma_{I\in\mathcal{I}}.(-)_I}{\underset{\bot}{\leftrightarrows}}} \mathcal{C}^{\mathcal{I}} \underset{\Pi_{I\in\mathcal{I}}.(-)_I}{\overset{\Delta}{\underset{\bot}{\leftrightarrows}}} \mathcal{C}$$

The limit preservation (1) and colimit reverse (2) continues to hold for dependent products and sums.

$$\Pi_{I\in\mathcal{I}}\mathcal{C}(X, A(I)) \cong \mathcal{C}(X, \Pi_{I\in\mathcal{I}}A(I) \tag{3}$$
$$\mathcal{C}(\Sigma_{I\in\mathcal{I}}A(I), X) \cong \Pi_{I\in\mathcal{I}}\mathcal{C}(A(I), X) \tag{4}$$

2.2.1 Powers and CoPowers

Now we consider categories of constant functors $\mathcal{C}^{\mathcal{I}}$ and keep $\mathcal{I}$ as the discrete category. The limits and colimits of these functors are called powers and copowers which can be indicated by $\Sigma\mathcal{I}.A = \mathcal{I} \bullet A$ and $\Pi\mathcal{I}.B = B^{\mathcal{I}}$

$$\mathcal{C} \underset{\Delta}{\overset{\Sigma\mathcal{I}.(-)}{\underset{\bot}{\leftrightarrows}}} \mathcal{C}^{\mathcal{I}} \underset{\Pi\mathcal{I}.(-)}{\overset{\Delta}{\underset{\bot}{\leftrightarrows}}} \mathcal{C}$$

Now equations (3) and (4) in turn specialise to powers and copowers

$$\mathcal{C}(X, A)^{\mathcal{I}} \cong \mathcal{C}(X, A^{\mathcal{I}}) \tag{5}$$
$$\mathcal{C}(\mathcal{I} \bullet A, X) \cong \mathcal{C}(A, X)^{\mathcal{I}} \tag{6}$$

As a consequence of (5) and (6) we get that

$$\mathcal{C}(\mathcal{I} \bullet A, B) \cong \mathcal{C}(A, B)^{\mathcal{I}} \cong \mathcal{C}(A, B^{\mathcal{I}})$$

Now since $\mathcal{I}$ is the discrete category (it has no arrows) it can be regarded as a set! Hence, the set of natural transformation betweeen to constant functors is just the set of functions between the images of these indexed by $\mathcal{I}$

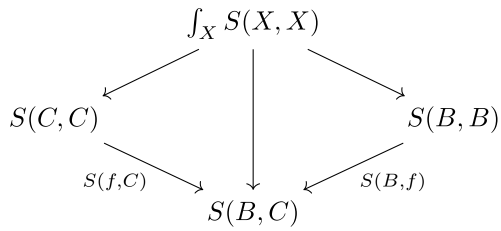
$$\mathcal{C}^{\mathcal{I}}(\Delta X, \Delta Y) \cong \mathcal{C}(X, Y)^{\mathcal{I}} \cong \mathcal{I} \to \mathcal{C}(X, Y) \tag{7}$$

Note that, since $\mathcal{I}$ and $\mathcal{C}(X, Y)$ are sets, then $\mathcal{C}(X, Y)^{\mathcal{I}}$ is the exponential object in **Set** and hence it is isomorphic to $\mathcal{I} \to \mathcal{C}(X, Y)$ which is the set of functions.

In **Set**, $\mathcal{I} \bullet A = A \times \mathcal{I}$ and $B^{\mathcal{I}}$ is the function space $\mathcal{I} \to B$.

3 Ends and Coends

Sometimes it useful to talk about limits and colimits of diagrams that have a contravariant component. These are called *ends* and *coends*. Consider a functor $S : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, the end of S is the limit of S when S is seeing as a diagram $S(C,C) \xrightarrow{S(f,C)} S(B,C) \xleftarrow{S(B,f)} S(B,B)$

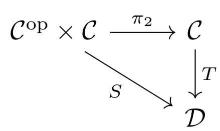


This is though not a very precise definition since the limit needs to be defined on a covariant functor. However, it can be shown [5, Chapter IX, Proposition 1], that for every category $\mathcal{C}$ and functor $S : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ there exists a category $\mathcal{C}^{\S}$ and functor $S^{\S} : \mathcal{C}^{\S} \to \mathcal{D}$ such that

$$\int_C S(C,C) \cong \varprojlim_C S^{\S} C \tag{8}$$

We refer the reader to the more in-depth presentations of this fact [5, 4].

Moreover, whenever S is "dummy" in the first variable, i.e. S factors through the second projection as in



then the end coincides with the limit of T

$$\int_{C:\mathcal{C}} S(C,C) = \varprojlim_{C:\mathcal{C}} TC \tag{9}$$

Ends behave similarly to universal quantification. For a functor $S : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{E}$

$$\int_{C:\mathcal{C}} \int_{D:\mathcal{D}} S(C,C,D,D) \cong \int_{D:\mathcal{C}} \int_{C:\mathcal{D}} S(C,C,D,D) \tag{10}$$

This property is known as the *exchange rule* which for integrals it corresponds to the Fubini rule [4].

3.1 Preservation of Ends

A functor $H : \mathcal{C} \to \mathcal{D}$ is said to *preserve the end* of a functor $S : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$

$$H \int_{C \in \mathcal{C}} S(C, C) = \int_{C \in \mathcal{C}} HS(C, C)$$

In other words, when $w : e \ddot{\to} S$ is an end of S and $Hw : He \ddot{\to} HS$ is an end for HS.

For example, as it was the case for the limits and colimits, the homset functor preserves ends in the following way

$$\mathcal{C}(X, \int_{C:\mathcal{C}} S(C, C)) = \int_{C:\mathcal{C}} (\mathcal{C}(X, S(C, C))) \tag{11}$$

and reserves ends into coends

$$\mathcal{C}(\int^{C:\mathcal{C}} S(C, C), X) = \int_{C:\mathcal{C}} (\mathcal{C}(S(C, C)), X) \tag{12}$$

3.2 Natural transformations and Ends

Natural transformations are examples of ends. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, the end of the homset functor $\mathcal{D}^{\mathcal{C}}(F-, G-)$ is the set of natural transformations from F to G

$$\text{Nat}(F, G) = \int_{C:\mathcal{C}} \mathcal{D}(FC, GC) \tag{13}$$

4 The Yoneda Lemma

Assume a locally small category $\mathcal{C}$, then for all covariant functors $F : \mathcal{C} \to \mathbf{Set}$,

$$FC \cong \mathcal{C}(C, -) \dot{\to} F \tag{14}$$

The functions witnessing the isomorphism are given by $\phi(x, g) = F(g)(x)$ and its inverse $\psi(i) = i_C(id_C)$. This lemma is sometimes stated for contravariant functors $F : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ as follows

$$FC \cong \mathcal{C}(-, C) \dot{\to} F \tag{15}$$

But notice since $\mathcal{C}(-, C)$ is equal to $\mathcal{C}^{\text{op}}(C, -)$, by definition of opposite category, this is just a special case of (14).

The homset functor $\mathcal{C}(-, -) : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ is also called the Yoneda embedding, y. The fact that y is an embedding means that the y is a fully faithful functor, i.e. it is bijection on morphisms, and therefore it preserves (because it is a functor) and reflects (because it is fully faithful) isomorphisms

$$y_C \cong y_D \iff C \cong D \tag{16}$$

Moreover,

$$y_C(f) = y_D(g) \iff f = g \tag{17}$$

$$y_C(f) = y_D(g) \iff f = g \tag{18}$$

4.1 The mini Yoneda Lemma for Type Theorists

Disclaimer: This section is taken from a lecture by Roy Crole at the Midlands Summer School 2018.

Say that you have a typed language with a unary constructor R which has the following typing rule

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathtt{R}(t) : B} \tag{19}$$

The task is to give a semantic interpretation $[\![\cdot]\!]$ for the language by induction on the typing judgment $\Gamma \vdash t : A$ such that terms are interpreted as morphisms $[\![\Gamma]\!] \xrightarrow{[\![t]\!]} [\![A]\!]$, assuming for course $[\![\cdot]\!]$ is also defined separately for contexts and types.

In order to interpret (19), as mentioned above, we do induction on the typing judgment so that, by induction, we know there exists a morphism $[\![\Gamma]\!] \xrightarrow{[\![t]\!]} [\![A]\!]$ and we have to construct a morphism $[\![\Gamma]\!] \xrightarrow{[\![\mathtt{R}(t)]\!]} [\![B]\!]$.

In the remainder of this section we remove the semantics brackets for simplicity, for example, assuming A be interpretation of $[\![A]\!]$, $t : \Gamma \to A$ the interpretation of t an so on.

In this situation it can be quite tricky sometimes to figure out what this moprhism should be since there is some plumming needed to pass around the context. A particular instantiation of the Yoneda lemma states that given a morphism $t : \Gamma \xrightarrow{t} A$ and a morphism $R : A \to B$ there is a canonical way to construct a morphism $\Gamma \xrightarrow{R(t)} B$.

To show this we instantiate the contravariant Yoneda lemma 15 by setting $F = \mathcal{C}(-, B)$. Then for all objects $A : \mathcal{C}^{\mathrm{op}}$ we have

$$\mathcal{C}(A, B) \cong \mathcal{C}(-, A) \stackrel{.}{\to} \mathcal{C}(-, B)$$

Let $R : A \to B$ be the interpretation of R then, one side of the isomorphism is $\phi(R, t) = F(t)(R) = \mathcal{C}([\![t]\!], B)(R)$. In other words, the interpretation of $R(t)$ is simply $R \circ t$.

4.2 Exponentials in Presheaf Categories

Say we are working in the presheaf category over $\mathcal{C}$, namely $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

We want to find out what the exponential in this category is. It may be tempting to define the exponential as the natural transformations, but this is not what is happening. If the exponential existed then it would be an object in $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ and at this point we would be able to apply the Yoneda lemma instantiating F with A^B in (14). We then calculate as follows using properties of ends and exponentials:

$$
\begin{aligned}
B^A(X) &\cong \{ \text{ Yoneda Lemma (14) } \} \\
&\quad \mathcal{C}(-,X) \dot{\to} B^A \\
&\quad \cong \{ \text{ Exponentials in functor categories } \} \\
&\quad \mathcal{C}(-,X) \times A \dot{\to} B \\
&\quad \cong \{ \text{ Ends as natural transformations (13) } \} \\
&\quad \int^{C:\mathcal{C}} \mathbf{Set}(\mathcal{C}(C,X) \times A(C), B(C))
\end{aligned}
$$

Thus the exponential at stage X is the set natural transformations from the functor A to the functor B, but restricted to the components C such that there exists at least one arrow $C \to X$.

5 Kan Extensions

Consider the category $\mathcal{C}$ formed by these objects and arrows

$$A \to C \leftarrow B$$

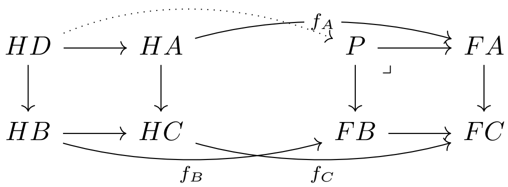
and the category $\mathcal{D}$ formed by the following objects and arrows

$$
\begin{array}{ccc}
D & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & C
\end{array}
$$

Clearly, $\mathcal{C}$ is a full subcategory of $\mathcal{D}$, indicated by the presence of the inclusion functor $i : \mathcal{C} \hookrightarrow \mathcal{D}$ since A, B and C contained in $\mathcal{D}$ along with their associated arrows. Now given a functor $F : \mathcal{C} \to \mathcal{E}$ how can we extend the functor to the category $\mathcal{D}$ such that this functor agrees with F on the objects and arrows that already belong to $\mathcal{C}$? Well notice that this functor needs to send D to an object GD which has arrows $GD \to FX = GX$ for each $X \in \mathcal{C}$ for which there is an arrow $D \to X$. A cone for the following diagram will do since D has an arrow into every object of $\mathcal{C}$, but since we will need to prove this object is the "maximal" one we choose the limiting cone of this diagram which is the pullback P

$$
\begin{array}{ccc}
P & \longrightarrow & FA \\
\downarrow & \lrcorner & \downarrow \\
FB & \longrightarrow & FC
\end{array}
$$

Now we need to ensure that this assignment is the "largest" one we could have picked. To do this we pick a generalised element of F, namely, for a functor $H : \mathcal{D} \to \mathcal{E}$ and for a generalised element $f : H \circ i \dot{\to} F$ we need to show there exists a unique map $H \dot{\to} G$ such that on objects in $\mathcal{C}$ this map agrees with f[1]. Now, for $X \in \mathcal{C}$, this is just f since $GX = FX$. We are left to show that there exists a map $HD \to GD = P$, but this is easy to see since HD is a cone for the functor F



Let us generalise this a bit further. Let $\mathcal{C} \hookrightarrow \mathcal{D}$ be a full subcategory of $\mathcal{D}$. Let $i : \mathcal{C} \to \mathcal{D}$ be the inclusion functor and let a functor $F : \mathcal{C} \to \mathcal{E}$. We want to *extend* this functor to a functor $G : \mathcal{D} \to \mathcal{E}$ such that for every object $X \in \mathcal{C}$, G sends X to FX.

Since G has to be a functor we need for every morphism $Y \to Y'$ in $\mathcal{D}$ to define what the functorial action of G is. Now let us restrict to the case when Y' is an $X \in \mathcal{C}$. When $Y \in \mathcal{D}$, but $Y \notin \mathcal{C}$, for every morphism $Y \to X$ in $\mathcal{D}$, the functorial action of G has to be of type $GY \to FX$, hence we define GY has the object that has morphisms into every object FX that has a map $Y \to X$. In particular, we want to take the universal such cone which is the limit of a functor $F \circ \pi_1 : Y/i \to \mathcal{E}^{\mathcal{C}}$ where Y/i is the comma category formed of triples $(Y \in \mathcal{D}, X \in \mathcal{C}, f : Y \to iX)$ and π is the projection functor $\pi : Y/i \to \mathcal{C}$

$$GY = \varprojlim_{(Y,X,f:Y\to iX)} (F \circ \pi)(Y, X, f : Y \to iX) = \varprojlim_{(Y,X,f:Y\to X)} FX$$

By (9) and since this functor is covariant this limit is isomorphic to the following end formula

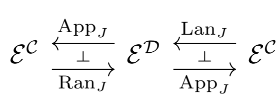
$$GY \cong \int_{(Y,X,f:Y\to X)} FX$$

which is the end of the functor $F \circ \pi \circ \pi_2$.

5.1 Kan Extensions

Consider a reindexing functor $J : \mathcal{C} \to \mathcal{D}$ and define the functor $- \circ J : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$ also named App_J.

We want to find out whether App_J has a left and right adjoint. We call its left and right adjoint Lan_J and Ran_J respectively.



[1]For the attentive reader, this conditions is the universal property of the counit required for G to be the right adjoint to the functor $- \circ i$

Now because Lan_J is the left adjoint and Ran_J to be the right one we would expect the following to be a natural isomorphisms

$$\mathcal{E}^{\mathcal{D}}(\mathrm{Lan}_J H, G) \cong \mathcal{E}^{\mathcal{C}}(H, \mathrm{App}_J G) \qquad \mathcal{E}^{\mathcal{C}}(\mathrm{App}_J H, G) \cong \mathcal{E}^{\mathcal{D}}(H, \mathrm{Ran}_J G)$$

In the MacLane [5] he is defining the right Kan extension and the proving it is the right adjoint. Here we take a different approach. By using the Yoneda lemma we derive the right adjoint which is unique up to isomorphism so it must be the right Kan extension. This proof is taken form Hinze's work on generic programming [3]. Here I have made it a bit more precise.

$$\begin{aligned}
\mathcal{E}^{\mathcal{C}}(\mathrm{App}_J A, B) = \{ & \text{ Homsets in the exponential category are natural transformations } \} \\
& \mathrm{Nat}(\mathrm{App}_J A, B) \\
& \cong \{ \text{ Natural transformations are ends (13) } \} \\
& \int_{X:\mathcal{C}} \mathcal{C}(AJX, BX) \\
& \cong \{ \text{ Yoneda with } \mathcal{C}(A-, BX) \ \} \\
& \int_{X:\mathcal{C}} \mathrm{Nat}(\mathcal{D}(-, JX), \mathcal{C}(A-, BX)) \\
& \cong \{ \text{ by (13) } \} \\
& \int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{D}(Y, JX) \to \mathcal{C}(AY, BX) \\
& \cong \{ \text{ by (7) } \} \\
& \int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{C}(AY, BX)^{\mathcal{D}(Y,JX)} \\
& \cong \{ \text{ by (5) } \} \\
& \int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{C}(AY, BX^{\mathcal{D}(Y,JX)}) \\
& \cong \{ \text{ by (10) } \} \\
& \int_{Y:\mathcal{D}} \int_{X:\mathcal{C}} \mathcal{C}(AY, BX^{\mathcal{D}(Y,JX)}) \\
& \cong \{ \text{ Homsets preserve ends (11) } \} \\
& \int_{Y:\mathcal{D}} \mathcal{C}(AY, \int_{X:\mathcal{C}} BX^{\mathcal{D}(Y,JX)}) \\
& \cong \{ \text{ Natural transformation are ends (13) } \} \\
& \mathrm{Nat}(A-, \int_{X:\mathcal{C}} BX^{\mathcal{D}(-,JX)}) \\
& \cong \{ \text{ Homsets in the exponential category are natural transformations } \} \\
& \mathcal{E}^{\mathcal{D}}(A-, \int_{X:\mathcal{C}} BX^{\mathcal{D}(-,JX)})
\end{aligned}$$

For all functors $J : \mathcal{C} \to \mathcal{D}$, $A : \mathcal{C} \to \mathcal{E}$ and $B : \mathcal{D} \to \mathcal{E}$

$$\mathrm{Ran}_J AY = \int_{X \in \mathcal{C}} \Pi_{\mathcal{D}(Y,JX)} AX$$

We now compute the left Kan extension. I could not find this proof is not in [3, 5].

$$\begin{aligned}
\mathcal{E}^{\mathcal{C}}(A, \mathrm{App}_J B) &= A \dot{\to} \mathrm{App}_J B \\
&\cong \{ \text{ Natural transformations are ends (13) } \} \\
&\int_{X:\mathcal{C}} \mathcal{C}(AX, BJX) \\
&\cong \{ \text{ by Yoneda with } \mathcal{C}(AX, B-) \} \\
&\int_{X:\mathcal{C}} \mathcal{D}(JX, -) \dot{\to} \mathcal{C}(AX, B-) \\
&\cong \{ \text{ Natural transformations are ends(13) } \} \\
&\int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{D}(JX, Y) \to \mathcal{C}(AX, BY) \\
&\cong \{ \text{ The set functions space is a power (7) } \} \\
&\int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{C}(AX, BY)^{\mathcal{D}(JX,Y)} \\
&\cong \{ \text{ Homsets revert powers into copowers (6) } \} \\
&\int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{C}(\mathcal{D}(JX, Y) \bullet AX, BY) \\
&\cong \{ \text{ Switching over ends (10) } \} \\
&\int_{Y:\mathcal{D}} \int_{X:\mathcal{C}} \mathcal{C}(\mathcal{D}(JX, Y) \bullet AX, BY) \\
&\cong \{ \text{ Homsets reverse ends into coends (12) } \} \\
&\int_{Y:\mathcal{D}} \mathcal{C}(\int^{X:\mathcal{C}} \mathcal{D}(JX, Y) \bullet AX, BY) \\
&\cong \{ \text{ Natural transformations are ends (13) } \} \\
&\mathrm{Nat}(\int^{X:\mathcal{C}} \mathcal{D}(JX, -) \bullet AX, B) \\
&= \mathcal{E}^{\mathcal{D}}(\int^{X:\mathcal{C}} \mathcal{D}(JX, -) \bullet AX, B)
\end{aligned}$$

Now by looking at what we have got we define the left Kan extension as a functor parametrised by A

$$\mathrm{Lan}_J AY = \int^{X \in \mathcal{C}} \mathcal{D}(JX, Y) \bullet AX$$

5.2 Yoneda revisited

5.2.1 Yoneda

Given a functor $F : \mathcal{C} \to \mathbf{Set}$, the Yoneda lemma says that F is isomorphic to the right Kan extension over the identity functor of F

$$F \cong \mathrm{Ran}_{\mathrm{Id}} F \tag{20}$$

We compute as follows

$$\begin{aligned} FC &\cong \mathrm{Nat}(\mathcal{C}(C, -), F) \\ &\cong \int_{X:\mathcal{C}} \mathbf{Set}(\mathcal{C}(C, X), FX) \\ &\cong \int_{X:\mathcal{C}} \mathcal{C}(C, X) \to FX \\ &\cong \int_{X:\mathcal{C}} FX^{\mathcal{C}(C,X)} \\ &\cong \mathrm{Ran}_{\mathrm{Id}} FC \end{aligned}$$

Another way of proving this is by using the fact that $\mathrm{App}_{\mathrm{Id}}$ is left adjoint to $\mathrm{Ran}_{\mathrm{Id}}$.

$$\mathcal{E}^{\mathcal{C}}(\mathrm{App}_{\mathrm{Id}} G, F) = \mathcal{E}^{\mathcal{C}}(G, F) \cong \mathcal{E}^{\mathcal{C}}(G, \mathrm{Ran}_{\mathrm{Id}} F)$$

Thus, since $\mathcal{E}^{\mathcal{C}}(-, F) \cong \mathcal{E}^{\mathcal{C}}(-, \mathrm{Ran}_{\mathrm{Id}} F)$ by Yoneda (16) we have $F \cong \mathrm{Ran}_{\mathrm{Id}} F$.

5.2.2 CoYoneda

The coYoneda lemma states that

$$F \cong \mathrm{Lan}_{\mathrm{Id}} F \tag{21}$$

This can be proven by instantiating the left Kan extension with the identity functor obtaining the adjunction $\mathrm{Lan}_{\mathrm{Id}} \dashv \mathrm{App}_{\mathrm{Id}}$. From the adjunction we know that

$$\mathcal{E}^{\mathcal{C}}(\mathrm{Lan}_{\mathrm{Id}} F, G) \cong \mathcal{E}^{\mathcal{C}}(F, \mathrm{App}_{\mathrm{Id}} G) = \mathcal{E}^{\mathcal{C}}(F, G \circ \mathrm{Id}) = \mathcal{E}^{\mathcal{C}}(F, G)$$

is a natural in F and G. Because this isomorphism is natural we know that $\mathcal{E}^{\mathcal{C}}(\mathrm{Lan}_{\mathrm{Id}} F, -) \cong \mathcal{E}^{\mathcal{C}}(F, -)$ which by (16) implies $\mathrm{Lan}_{\mathrm{Id}} F \cong F$ since $\mathcal{E}^{\mathcal{C}}(-, -)$ is the Yoneda embedding.

5.3 Left and Right Shifts

Left and right shifts are just particular cases of Kan extensions where $\mathcal{C}$ is 1. So now the reindexing functor is just an object $D : 1 \to \mathcal{D}$. If we now start abusing some notation setting D to mean $D(*)$ we have $\mathrm{App}_D H = H \circ D = HD(*) = HD$

$$\mathcal{E} \underset{\mathrm{Rsh}_D}{\overset{-D}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{E}^{\mathcal{D}} \underset{-D}{\overset{\mathrm{Lsh}_D}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{E}$$

At this point the natural isomoprhisms induces by the adjunctions are as follows

$$\mathcal{E}^{\mathcal{D}}(\mathrm{Lsh}_D F, G) \cong \mathcal{E}(F, GD) \qquad \mathcal{E}(FD, G) \cong \mathcal{E}^{\mathcal{D}}(F, \mathrm{Rsh}_J G)$$

What is interesting to note is that left and right Kan extensions simplify into left and right shifts

$$\mathrm{Lsh}_D FY = \mathcal{D}(D, Y) \bullet F \qquad \mathrm{Rsh}_D GY = G^{\mathcal{D}(Y,D)}$$

6 Monads from Kan Extensions

6.1 The Codensity Monad

The codensity monad is just the right Kan extension of J along J

$$\mathrm{Cod}\ JX = \mathrm{Ran}_J JX = \int_{Y:\mathcal{C}} JY^{\mathcal{D}(X,JY)}$$

6.2 The Codensity Transformation

If $L \dashv R$ then both $L \circ - \vdash R \circ -$ and $- \circ R \vdash - \circ L$ are adjunctions.

$$\text{If} \quad \mathcal{L} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{R} \quad \text{then} \quad \mathcal{E}^{\mathcal{L}} \underset{-\circ L}{\overset{-\circ R}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{E}^{\mathcal{R}}$$

Because of this fact there is a natural isomorphism

$$\mathcal{E}^{\mathcal{L}}(F \circ R, G) \cong \mathcal{E}^{\mathcal{R}}(F, G \circ L)$$

Now, since $F \circ R$ is $\mathrm{App}_R F$ then $\mathcal{E}^{\mathcal{L}}(F \circ R, G) \cong \mathcal{E}^{\mathcal{R}}(F, \mathrm{Ran}_R G)$. But then we know also that

$$\mathcal{E}^{\mathcal{L}}(F, G \circ L) \cong \mathcal{E}^{\mathcal{R}}(F, \mathrm{Ran}_R G)$$

Since the Yoneda embedding $\mathcal{E}^{\mathcal{R}}(-,-) : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is fully faithful, by (16) then $G \circ L \cong \mathrm{Ran}_R G$. (But this is also the proof that adjoints are unique up-to isomorphism). Now, if $G = R$ then we get that the monad

$$R \circ L \cong \mathrm{Ran}_R R \tag{22}$$

7 On Free Monads

Given an endofunctor $F : \mathcal{C} \to \mathcal{C}$ the free monad M over F is the monad freely generated by the constructors $\eta : \mathrm{Id} \to M$ and $\mathrm{op} : FM \to M$. The fact that is a monad implies it must have a join operation as well $\mu : MM \to M$. This monad M is also indicated by F^*.

When it exists, the free monad is the least solution to the equation

$$F^*A \cong A + FF^*A$$

It can be shown that this is the free F-algebra in the sense that it is the monad arising from a free-forgetful adjunction

$$F\text{-Alg} \underset{U}{\overset{\text{Free}}{\underset{\bot}{\leftrightarrows}}} \mathcal{C} \circlearrowleft F*$$

where $F^* = U\mathrm{Free}$.

The ceiling and floor witness the natural isomorphism on both A and B

$$\lfloor\cdot\rfloor : F\text{-Alg}(\mathrm{Free}\ A, B) \cong \mathcal{C}(A, UB) : \lceil\cdot\rceil$$

As usual, the unit of the monad is given by $\eta = \lfloor id_{\mathrm{Free}A}\rfloor$ and the multiplication is derived by using the counit $\mu = U\epsilon_{\mathrm{Free}}$. Now the map $FF^*A \to F^*A$ is the F-algebra given by Free$A$. To see this let F^*A to be UFree$A$ (it's just a name). Say that Free takes an object $A : \mathcal{C}$ an sends it to an F-algebra, Free$A = (X, \mathrm{alg} : FX \to X)$ for some X and some algebras alg. But that means that $U\mathrm{Free}A = X$ which implies X is F^*A and that alg is the map $\mathrm{op} : FF^*A \to F^*A$ we were looking for.

7.1 Free Monads and The Right Kan Extension

From the previous section we know that every monad arising from an adjunction $L \dashv R$ (hence every monad!) is isomorphic to the right Kan extension of R along R, denoted by $\mathrm{Ran}_R R$ and also known as the *codensity monad*.

Since the free monad is a monad, also the free monad can be transformed using the codensity transformation. Since the algebraically free monad factors as $F^* = U\mathrm{Free}$ then by (22)

$$F^* \cong \mathrm{Ran}_U U$$

By unfolding the definitions we get that

$$F^*A \cong \int_{Z:F\text{-Alg}} UZ^{\mathcal{C}(A,UZ)} \tag{23}$$

However, there is another way to transform the free monad. Using Yoneda (20)

$$F^*A \cong A + FF^*A \cong A + (\mathrm{Ran}_{\mathrm{Id}}F)F^*X \cong A + \int_{X:\mathcal{C}} FX^{\mathcal{C}(F^*A,X)} \tag{24}$$

7.2 Free Monads and the Left Kan Extension

Using CoYoneda (21)

$$F^*A \cong A + F \circ F^* \cong A + (\mathrm{Lan}_{\mathrm{Id}} F)F^*A \cong A + \int^{X:\mathbf{Set}} \mathcal{C}(X, F^*A) \bullet FX \quad (25)$$

In **Set**,

$$F^*A \cong A + \int_{X:\mathcal{C}} FX \times (X \to F^*A)$$

7.3 Algebras for the Left Kan extension

Every G-algebra is isomorphic to the algebras for the left Kan extension on G.

$$\begin{aligned}
\mathcal{D}(\mathrm{Lan}_{\mathrm{Id}} GA, A) &\cong \mathcal{D}^{\mathcal{D}}(\mathrm{Lan}_{\mathrm{Id}} G, \mathrm{Rsh}_A A) && \{ \text{ by } -A \dashv \mathrm{Rsh}_{\mathrm{Id}} \ \} \\
&\cong \mathcal{D}^{\mathcal{D}}(G, \mathrm{Rsh}_A A \circ \mathrm{Id}) && \{ \text{ by } \mathrm{Lan}_{\mathrm{Id}} \dashv \mathrm{App}_{\mathrm{Id}} \ \} \\
&\cong \mathcal{D}^{\mathcal{D}}(G, \mathrm{Rsh}_A A) \\
&\cong \mathcal{D}(GA, A) && \{ \text{ by } -A \dashv \mathrm{Rsh}_{\mathrm{Id}} \ \}
\end{aligned}$$

This proof is worth of reminding, but a simpler way to do it is to use CoYoneda (21) directly

$$\mathcal{D}(\mathrm{Lan}_{\mathrm{Id}} GA, A) \cong \mathcal{D}(GA, A)$$

7.4 The Freest Monad

Given a functor $J : \mathcal{C} \to \mathcal{D}$ and an endofunctor $F : \mathcal{C} \to \mathcal{D}$ the freest monad is defined as follows

$$F^{\mathrm{st}}_J A \cong JA + \mathrm{Lan}_J F(F^{\mathrm{st}} A)$$

Note that F is not an endofunctor and so the freest monad is in fact a *relative monad* [1]. The free monad over an endofunctor $F : \mathcal{C} \to \mathcal{C}$ is derivable by setting J to the identity functor

$$F^*A = F^{\mathrm{st}}_{\mathrm{Id}} A \cong \mathrm{Id}A + \mathrm{Lan}_{\mathrm{Id}} F(F^{\mathrm{st}} A) \cong A + F(F^{\mathrm{st}} A)$$

The last step is of course the coYoneda lemma (21). I am not sure yet if the freeest monad is derivable from the free monad.

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